

On the distribution of zeros of the derivative of Selberg's zeta function associated to finite volume Riemann surfaces

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Abstract

In 1934, A. Speiser proved that the Riemann hypothesis for the Riemann zeta function $\zeta_{\mathbb{Q}}$ is equivalent to the statement that its derivative $\zeta'_{\mathbb{Q}}(s)$ is non-vanishing in the half-plane $\operatorname{Re}(s) < 1/2$, see [56]. Further results in the study of the zeros of $\zeta'_{\mathbb{Q}}$ were established by Spira [57] and Levinson-Montgomery [42]. The theorems from [42] played an important role in Levinson's proof that at least one-third of all non-trivial zeros of the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ are located on the critical line $\operatorname{Re}(s) = 1/2$; see [41] for Levinson's paper as well as [52] for Selberg's classical article in this direction. With these results, it then becomes important to study the zeros of derivatives of zeta functions.

In [44], W. Luo investigated the distribution of zeros of the derivative of the Selberg zeta function associated to compact hyperbolic Riemann surfaces. In essence, the main results in [44], which were extended in [25], [26], [46] and [49], involve the following three points: Finiteness for the number of zeros in the half plane $\operatorname{Re}(s) < 1/2$; an asymptotic expansion for the counting function measuring the vertical distribution of zeros; and an asymptotic expansion for the counting function measuring the horizontal distance of zeros from the critical line. In the present article, we study the more complicated setting of distribution of zeros of the derivative of the Selberg zeta function associated to a non-compact, finite volume hyperbolic Riemann surface M . There are numerous difficulties which exist in the non-compact case that are not present in the compact setting, beginning with fact that in the non-compact case the Selberg zeta function does not satisfy the analogue of the Riemann hypothesis. To be more specific, we actually study the zeros of $(Z_M H_M)'$, where Z_M is the Selberg zeta function and H_M is the Dirichlet series component of the scattering matrix, both associated to an arbitrary finite-volume hyperbolic Riemann surface M . As in the above mentioned articles, our main results address finiteness of zeros in the half plane $\operatorname{Re}(s) < 1/2$, an asymptotic count for the vertical distribution of zeros, and an asymptotic count for the horizontal distance of zeros.

Generally speaking, the philosophy behind the Phillips-Sarnak conjecture suggests that the spectral analysis of the Laplacian acting on smooth functions on M should depend on the arithmetic nature of the underlying Fuchsian group Γ . One realization of the spectral analysis of the Laplacian is the location of zeros of Z_M . Our analysis yields an invariant A_M which appears in the vertical and horizontal distribution of zeros of $(Z_M H_M)'$, and we show that A_M has different values for surfaces associated to two topologically equivalent yet different arithmetically defined Fuchsian groups. We view this aspect of our main theorem as both supporting the Phillips-Sarnak philosophy and indicating the existence of further spectral phenomena which provides a additional refinement within the set of arithmetically defined Fuchsian groups.

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1 Introduction

1.1 Classical results for the Riemann zeta function

In [56], A. Speiser proved that the Riemann hypothesis for the Riemann zeta function $\zeta_{\mathbb{Q}}$ is equivalent to proving that its derivative $\zeta'_{\mathbb{Q}}(s)$ is non-vanishing for $\text{Re}(s) < 1/2$. As a consequence of Speiser's theorem, and recognizing the Riemann hypothesis as a mathematical question of unparalleled historical and mathematical significance, one now recognizes the need to study the zeros of $\zeta'_{\mathbb{Q}}$, the derivative of the Riemann zeta function. The most ambitious goal would be to obtain a precise description of the location of the zeros of $\zeta'_{\mathbb{Q}}$, analogous to the Riemann hypothesis itself for $\zeta_{\mathbb{Q}}$. At this time, a more realistic goal would be to investigate distributional questions associated to the zeros of $\zeta'_{\mathbb{Q}}$, with or without employing underlying assumptions or hypotheses associated to the location of zeros of $\zeta_{\mathbb{Q}}$, such as the classical Riemann hypothesis.

Let $N_{\text{ver}}(T; \zeta_{\mathbb{Q}})$ denote the number of zeros $s = \sigma + it$ of the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ such that $\sigma \in (0, 1)$ and $0 < t < T$. Classically, it is known that

$$N_{\text{ver}}(T; \zeta_{\mathbb{Q}}) = \frac{T}{2\pi} \log(T/2\pi) - \frac{T}{2\pi} + O(\log T), \quad \text{as } T \rightarrow \infty. \quad (1)$$

In [7], vertical counting function $N_{\text{ver}}(T; \zeta'_{\mathbb{Q}})$ of non-trivial zeros of $\zeta'_{\mathbb{Q}}$ is studied; it is shown that

$$N_{\text{ver}}(T; \zeta'_{\mathbb{Q}}) = \frac{T}{2\pi} \log(T/2\pi) - (1 + \log 2) \frac{T}{2\pi} + O(\log T), \quad \text{as } T \rightarrow \infty. \quad (2)$$

As noted in [42], the above results can be combined to yield

$$N_{\text{ver}}(T; \zeta_{\mathbb{Q}}) = N_{\text{ver}}(T; \zeta'_{\mathbb{Q}}) + T \cdot \frac{\log 2}{2\pi} + O(\log T), \quad \text{as } T \rightarrow \infty.$$

Similar results for higher derivatives $\zeta_{\mathbb{Q}}^{(k)}$ of the Riemann zeta function are also given in [7].

The counting function

$$N_{\text{hor}}(T; \zeta'_{\mathbb{Q}}) = \sum_{\substack{\zeta'_{\mathbb{Q}}(\sigma+it)=0 \\ 0 < t < T, 1/2 < \sigma < 1}} (\sigma - 1/2)$$

is defined to study the horizontal distribution of the zeros of $\zeta'_{\mathbb{Q}}$. From [42], Theorem 5. it is easy to deduce that

$$N_{\text{hor}}(T; \zeta'_{\mathbb{Q}}) \simeq \frac{T}{2\pi} \log \log(T/2\pi), \quad \text{as } T \rightarrow \infty. \quad (3)$$

Obviously, the Riemann hypothesis addresses the behavior of the horizontal counting function $N_{\text{hor}}(T; \zeta_{\mathbb{Q}})$, asserting that the function is identically zero.

In [41], N. Levinson used (2) to prove that at least one-third of the non-trivial zeros of $\zeta_{\mathbb{Q}}(s)$ lie on the critical line $\text{Re}(s) = 1/2$. Refined studies involving the horizontal distribution (3) are given in [55], with additional results in the following articles: [12], [19], [27], [39], [50] and [61], to name a few.

Within the field of random matrix theory, there are many points where the zeros of the Riemann zeta function arise: see the survey article [16] and references therein. Beyond this, the article [45] introduces a connection between zeros of $\zeta'_{\mathbb{Q}}$ and random matrix theory, thus further highlighting the results (2) and (3). Additional investigations into the zeros of the derivative of the Riemann zeta function continue, with new results appearing frequently: see [8], [17] and [18].

1.2 Selberg zeta functions for compact Riemann surfaces

Naturally, a generalization of (1), (2) and (3) can be considered for any zeta and L -function arising from number theory or elsewhere. In [44], W. Luo initiated the study of the zeros of the derivative Z'_M of the Selberg zeta function Z_M associated to a compact, hyperbolic Riemann surface M , proving

analogues of (2) and (3). Further refinements of the horizontal and vertical counting functions $N_{\text{hor}}(T; Z'_M)$ and $N_{\text{ver}}(T; Z'_M)$ were established in [25] and [26]. Let us summarize the three main results, which are location, vertical distribution, and horizontal distribution of the zeros of Z'_M .

In [44] it is shown that $Z'_M(s)$ has at most a finite number of non-trivial zeros in the half-plane $\text{Re}(s) < 1/2$. This result was strengthened in [46] and [49] where it is proved that $Z'_M(s)$ has no non-trivial zeros in the half-plane $\text{Re}(s) < 1/2$.

Let $\text{vol}(M)$ denote the hyperbolic volume of M . Let $\ell_{M,0}$ denote the length of the shortest closed geodesic on M ; a shortest closed geodesic on M sometimes is called a *systole* of M . Let $m_{M,0}$ denote the number of inconjugate geodesics whose length is $\ell_{M,0}$. Let $N_{\text{ver}}(T; Z'_M)$ denote the number of non-trivial zeros of $Z'_M(s)$ with height bounded by T ; in other words, where $s = \sigma + it$ with $\sigma \geq 1/2$ and $0 < t < T$. Extending the results in [44], it is proved in [25] and [26] that

$$N_{\text{ver}}(T; Z'_M) = \frac{\text{vol}(M)}{4\pi} T^2 - \frac{\ell_{M,0}}{2\pi} T + o(T) \quad \text{as } T \rightarrow \infty. \quad (4)$$

Let

$$N_{\text{hor}}(T; Z'_M) = \sum_{\substack{Z'_M(\sigma+it)=0 \\ 0 < t < T, \sigma > 1/2}} (\sigma - 1/2)$$

Then, building on the results from [44], it is proved in [25] and [26] that

$$N_{\text{hor}}(T; Z'_M) = \frac{T}{2\pi} \log T + \frac{T}{2\pi} \left(\frac{1}{2} \ell_{M,0} + \log \left(\frac{\text{vol}(M)(1 - \exp(-\ell_{M,0}))}{m_{M,0} \ell_{M,0}} \right) - 1 \right) + o(T) \quad \text{as } T \rightarrow \infty. \quad (5)$$

The study of the zeros of Z'_M is of particular interest because of the connection with spectral analysis. Recall that if s is a non-trivial zero of $Z_M(s)$, then $\lambda = s(1-s)$ is an eigenvalue of an L^2 -eigenfunction of the hyperbolic Laplacian which acts on the space of smooth functions on M . Common zeros of Z_M and Z'_M are multi-zeros of $Z_M(s)$, which, for non-trivial zeros of Z_M , correspond to multi-dimensional eigenspaces of the Laplacian. As shown on page 1143 of [44], all zeros of $Z'_M(s)$ on the line $\text{Re}(s) = 1/2$, except possibly at $s = 1/2$, correspond to multiple zeros of Z_M . The problem of obtaining non-trivial bounds for the dimension of eigenspaces of the Laplacian is very difficult; see page 160 of [34]. It is possible that refined information regarding (4) could possibly shed light on this important, outstanding question.

1.3 Non-compact Riemann surfaces

Let \mathbb{H} denote the hyperbolic upper half plane. Let $\Gamma \subseteq \text{PSL}_2(\mathbb{R})$ be any Fuchsian group of the first kind acting by fractional linear transformations on \mathbb{H} , and let M be the quotient space $\Gamma \backslash \mathbb{H}$. The question of studying the zeros of Z'_M when M is not compact begins with one possible difficulty stemming from the structure of the Selberg zeta function. As stated on page 498 of [29] as well as page 48 of [60], the Selberg zeta function itself has an infinite of zeros in the half-plane $\text{Re}(s) < 1/2$, namely at all the points where $\phi_M(s)$, the determinant of the scattering matrix, has poles. As we see with the Selberg zeta function for compact surfaces, and with the Riemann zeta function assuming the Riemann hypothesis, it seems necessary to study a function which itself has only trivial zeros in the left-half plane $\text{Re}(s) < 1/2$. It is from this point that the analysis of the present paper begins.

The function $\phi_M(s)$ has a decomposition into a product of a general Dirichlet series and Gamma functions. Specifically, from [34], [29] or [60], we can write

$$\phi_M(s) = \pi^{\frac{n_1}{2}} \left(\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^{n_1} \sum_{n=1}^{\infty} \frac{d(n)}{\mathfrak{g}_n^{2s}}$$

where n_1 is the number of cusps of M , and $\{d(n)\}$ and $\{\mathfrak{g}_n\}$ are sequences of real numbers with

$$0 < \mathfrak{g}_1 < \dots < \mathfrak{g}_n < \mathfrak{g}_{n+1} < \dots;$$

see also page 33, Theorem 1.5.3 of [20]. Let us write

$$\phi_M(s) = K_M(s) \cdot H_M(s)$$

where

$$K_M(s) = \pi^{\frac{n_1}{2}} \left(\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^{n_1} e^{c_1 s + c_2} \quad \text{with } c_1 = -2 \log \mathfrak{g}_1 \quad \text{and } c_2 = \log d(1), \quad (6)$$

and

$$H_M(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{r_n^{2s}} \quad \text{with } r_n = \mathfrak{g}_n / \mathfrak{g}_1 > 1 \quad \text{and } a(n) = d(n) / d(1). \quad (7)$$

The Dirichlet series expansion for $H_M(s)$ converges for all $\text{Re}(s) > 1$. We call the function H_M the Dirichlet series portion of the scattering determinant ϕ_M .

In general, the function H_M can be expressed as the determinant of a matrix whose entries are general Kloosterman sums; see, for example, Theorem 3.4, page 60 of [34] as well as Chapter 4 of [33]. The constants \mathfrak{g}_1 and \mathfrak{g}_2 are explained in terms of the left lower entries of the matrices appearing in the double coset decomposition of Γ . Therefore, the constants \mathfrak{g}_1 and \mathfrak{g}_2 are precisely connected to the Fuchsian group Γ .

Since ϕ_M satisfies the function equation $\phi_M(s)\phi_M(1-s) = 1$, the zeros and poles of ϕ_M are symmetrically located about the critical line $\text{Re}(s) = 1/2$. Furthermore, from Theorem 5.3, page 498 of [29], one has that $Z_M \phi_M$ has no non-trivial zeros in the half-plane $\text{Re}(s) < 1/2$. Consequently, the function $Z_M H_M$ has no non-trivial zeros in the half-plane $\text{Re}(s) < 1/2$. In fact, $(Z_M H_M)(s)$ is a holomorphic function on $\mathbb{C} \setminus (-\infty, 1/2]$.

As a result, rather than study zeros of Z'_M , we shall study the zeros of $(Z_M H_M)'$.

1.4 The main result

The function H'_M / H_M has admits the general Dirichlet series expansion

$$\frac{H'_M}{H_M}(s) = \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s}, \quad (8)$$

where series on the right converges absolutely and uniformly for $\text{Re}(s) \geq \sigma_0 + \epsilon > \sigma_0$ for sufficiently large σ_0 and where $\{q_i\}$ is a non-decreasing sequence of positive real numbers consisting of all finite products of numbers $r_n^2 > 1$. Obviously, $q_2 > q_1 = \inf q_i = (\mathfrak{g}_2 / \mathfrak{g}_1)^2$. Furthermore,

$$b(q_1) = -a(2) \log q_1 = -2(d(2)/d(1)) \log(\mathfrak{g}_2 / \mathfrak{g}_1).$$

Let $\ell_{M,0}$ be the length of a shortest closed geodesic, or systole, on M . With our notation from above, let

$$A_M = \min \left\{ e^{\ell_{M,0}}, (\mathfrak{g}_2 / \mathfrak{g}_1)^2 \right\}. \quad (9)$$

Here, we have dropped the subscript M on $(\mathfrak{g}_2 / \mathfrak{g}_1)^2$ in order to ease the notation; however, it is clear that $(\mathfrak{g}_2 / \mathfrak{g}_1)^2$ depends on M . Let $m_{M,0}$ denote the number of inconjugate closed geodesics on M with length $\ell_{M,0}$. If $e^{\ell_{M,0}} \neq (\mathfrak{g}_2 / \mathfrak{g}_1)^2$, let

$$a_M = \begin{cases} \frac{m_{M,0} \ell_{M,0}}{1 - e^{-\ell_{M,0}}}; & \text{if } e^{\ell_{M,0}} < (\mathfrak{g}_2 / \mathfrak{g}_1)^2 \\ b((\mathfrak{g}_2 / \mathfrak{g}_1)^2); & \text{if } e^{\ell_{M,0}} > (\mathfrak{g}_2 / \mathfrak{g}_1)^2 \end{cases}. \quad (10)$$

If $e^{\ell_{M,0}} = (\mathfrak{g}_2 / \mathfrak{g}_1)^2$, let

$$a_M = \frac{m_{M,0}\ell_{M,0}}{1 - e^{-\ell_{M,0}}} + b((\mathfrak{g}_2/\mathfrak{g}_1)^2). \quad (11)$$

Observe that a_M is the sum of the two terms which appear in the two cases in (10), not the arithmetic average as one would expect from elementary Fourier analysis.

With all this, the main result of this article is the following.

Theorem. *Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be any Fuchsian group of the first kind acting by fractional linear transformations on \mathbb{H} , and let M be the quotient space $\Gamma \backslash \mathbb{H}$. Let $Z_M(s)$ be the associated Selberg zeta function, and $H_M(s)$ be the Dirichlet series portion of the determinant of the associated scattering matrix.*

a) There are a finite number of non-trivial zeros of $(Z_M H_M)'(s)$ in the half-plane $\mathrm{Re}(s) < 1/2$. In addition, there exist some $t_0 > 0$ such that any zero of $(Z_M H_M)'(s)$ on the line $\mathrm{Re}(s) = 1/2$ with property $|\mathrm{Im}(s)| > t_0$ arises from a multiple zero of $Z_M(s)$.

b) Let us define the vertical counting function

$$N_{\mathrm{ver}}(T; (Z_M H_M)') = \#\{\rho = \sigma + it \mid (Z_M H_M)'(\rho) = 0 \text{ with } 0 < t < T\}.$$

Then

$$N_{\mathrm{ver}}(T; (Z_M H_M)') = \frac{\mathrm{vol}(M)}{4\pi} T^2 - \frac{T}{2\pi} (\log A_M + 2n_1 \log 2 + 2 \log \mathfrak{g}_1) + o(T), \text{ as } T \rightarrow \infty.$$

In particular, if M is co-compact, then (4) holds true.

c) Let us define the horizontal counting function

$$N_{\mathrm{hor}}(T; (Z_M H_M)') = \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ 0 < t < T \text{ and } \sigma > 1/2}} (\sigma - 1/2).$$

Then

$$\begin{aligned} N_{\mathrm{hor}}(T; (Z_M H_M)') &= \left(\frac{n_1}{2} + 1\right) \frac{T \log T}{2\pi} + \frac{T}{2\pi} \left(\log \frac{\mathrm{vol}(M) A_M^{1/2}}{|a_M|} - 1 \right) \\ &\quad + \frac{T}{2\pi} \left(\log \left(\frac{\mathfrak{g}_1}{\pi^{n_1/2} |d(1)|} \right) - \frac{n_1}{2} \right) + o(T), \text{ as } T \rightarrow \infty. \end{aligned}$$

In particular, if M is co-compact, then (5) holds true.

As stated in the Theorem, the above asymptotic formulas specialize in the case M is compact to give the main results in [44], [25], [26], [46], [47] and [49]. Similar results for zeros of higher derivatives of $Z_M H_M$ are presented in a later section. In addition, corollaries of the main theorem, analogous to results from [42], are derived.

1.5 Remarks concerning the Main Theorem

Aspects of the spectral analysis of the Laplacian acting on smooth functions on a hyperbolic Riemann surface can be measured by studying the zeros of the Selberg zeta function. As we discussed above, one equivalently can study the zeros of $Z_M H_M$. Indeed, the Selberg zeta function can be constructed using its divisor, which comes from the eigenvalues of the Laplacian and poles of the scattering matrix (see page 498 of [29]) together with general characterizing properties associated to its asymptotic behavior as $\mathrm{Re}(s) \rightarrow +\infty$. Therefore, by slight extension, the zeros of $(Z_M H_M)'$ provide another measure, in a sense, of the spectral analysis of the Laplacian. In this regard, the quantity $(\mathfrak{g}_2/\mathfrak{g}_1)^2$ is a new spectral invariant. Additionally, our Main Theorem indicates that for any given surface, the spectral analysis depends on the comparison of $e^{\ell_{M,0}}$ and $(\mathfrak{g}_2/\mathfrak{g}_1)^2$.

In section 7, we will show that for congruence subgroups one has the inequality $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$. However, this inequality does not hold for all arithmetic groups. We examine in detail the two "moonshine groups" $\Gamma_0^+(5)$ and $\Gamma_0^+(6)$. These two groups are arithmetic and have the same topological signature. However, for $\Gamma_0^+(5)$, we have that $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ whereas for $\Gamma_0^+(6)$ we have that $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$. We find it very interesting that, in the sense of our Main Theorem, not all arithmetic surfaces, even those with the same topological signature, have the same behavior.

Also in section 7, we argue that if one considers a degenerating family of hyperbolic Riemann surfaces within the moduli space of surfaces of fixed topological type, one eventually has the inequality $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ near the boundary. As a result, if one begins with congruence group and degenerates the corresponding surface, one will ultimately encounter a surface where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. More generally, however, it seems as if moduli space can be separated into sets defined by the sign of $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2$ where most, but not all, arithmetic surfaces are in the component where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 > 0$, and the Deligne-Mumford boundary lies in the component where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 < 0$.

We could not explicitly construct a surface where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 = 0$, even though we prove, in section 7, that such surfaces exist.

1.6 A comparison of counting functions

On page 456 of [29], D. Hejhal establishes the asymptotic behavior of the horizontal distribution of zeros of ϕ_M within the critical strip. In our notation, the zeros of ϕ_M within the critical strip coincide with the zeros of the Dirichlet series H_M , so then Theorem 2.22, page 456 of [29] establishes the asymptotic behavior of the horizontal counting function $N_{\text{hor}}(T; H_M)$.

Let M be any finite volume hyperbolic Riemann surface. We claim there exist a co-compact hyperbolic Riemann surface \tilde{M} such that $\text{vol}(M) = \text{vol}(\tilde{M})$, $\ell_{M,0} = \ell_{\tilde{M},0}$ and $m_{M,0} = m_{\tilde{M},0}$, which we argue as follows. In the case when the number n_1 of cusps of the surface M is even, we choose the surface \tilde{M}_1 to be any co-compact surface with genus $g_{\tilde{M}} = g_M + n_1/2$ and the same structure of elliptic points as M , hence $\text{vol}(M) = \text{vol}(\tilde{M}_1)$. If the number of cusps of the surface M is odd, we choose the surface \tilde{M}_1 to be any co-compact surface with genus $g_{\tilde{M}} = g_M + (n_1 - 1)/2$ such that it has the same structure of elliptic points as M , plus one additional elliptic point of order 2. By the Gauss-Bonnet formula, $\text{vol}(M) = \text{vol}(\tilde{M}_1)$. We then deform the surface \tilde{M}_1 in moduli space so that its shortest geodesic has the length equal to $\ell_{M,0}$ and the number of inconjugate geodesics of length $\ell_{M,0}$ is $m_{M,0}$.

Assume that M is such that, in the notation of (9), $A_M = \exp(\ell_{M,0})$. Then, as we will prove in a later section, one can combine Hejhal's theorem regarding $N_{\text{hor}}(T; H_M)$ with part (c) of the Main Theorem to establish the simple asymptotic relation

$$N_{\text{hor}}(T; (Z_M H_M)') = N_{\text{hor}}(T; Z_{\tilde{M}}') + N_{\text{hor}}(T; H_M) + o(T) \text{ as } T \rightarrow \infty. \quad (12)$$

In a later section, we will show that the relation (12) holds true when the derivative is replaced by the k th derivative, for all $k \geq 2$.

We find it very interesting that, in the case when M is such that $\exp(\ell_{M,0}) < (\mathfrak{g}_2/\mathfrak{g}_1)^2$, the coefficients of the first two terms, namely $T \log T$ and T , in the asymptotic development of the counting function $N_{\text{hor}}(T; (Z_M H_M)')$ coincide with known results, namely Hejhal's theorem and (5). The comparison (12) is vaguely reminiscent of the main result of [58]. In that article, the authors compute the curvature of a determinant line bundle on the moduli space of finite volume hyperbolic Riemann surfaces, showing that the curvature form consists of two parts: One part related to the curvature form in the compact case, and a second part defined using parabolic Eisenstein series.

1.7 Further comments

Weyl's law in its classical form evaluates the lead asymptotic behavior of the vertical counting function $N_{\text{ver}}(T; Z_M)$ for compact M . As far as is known, the expansion in T involves $\text{vol}(M)$ and

no other information associated to the uniformizing group Γ . If M is non-compact, the generalization of Weyl's law addresses the asymptotic behavior of

$$\#\{\lambda_{j,M} < 1/4 + T^2\} - \frac{1}{4\pi} \int_{-T}^T \phi'_M / \phi_M(1/2 + ir) dr \quad (13)$$

where $\lambda_{j,M}$ is the eigenvalue of an L^2 eigenfunction on M . The asymptotic expansion of (13) is recalled below (formula (87)) and, as in the compact case, all terms in the expansion involve elementary quantities associated to the uniformizing group Γ .

In section 9.1, we will express the function in (13) in terms of $N_{\text{ver}}(T; Z_M H_M)$, obtaining an expression which involves the constant \mathfrak{g}_1 . As a result, we accept the appearance of the term $\mathfrak{g}_2/\mathfrak{g}_1$ in our Main Theorem as being an appropriate generalization of a version of Weyl's law.

In a different direction, if one considers a degenerating family of finite volume hyperbolic Riemann surfaces, then it was shown in [31] that the asymptotic behavior of the associated sequence of vertical counting functions $N_{\text{ver}}(T; Z_M)$ has lead asymptotic behavior, for fixed T , which involves the lengths of the pinching geodesics; see Theorem 5.5 of [31]. As a result, we do not view the appearance of the invariant $\ell_{M,0}$ in (4) and (5) as a new feature when using Weyl's laws to understand refined information associated to the uniformizing group Γ .

However, we find the appearance of the constants A_M and a_M , as defined in (9), (10) and (11) to be surprising. In particular, for any given surface M , we do not know if there are conditions which will determine the value taken by A_M . If $\Gamma = \Gamma_0(N)$, a congruence subgroup, then we show that $A_M = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. If $\Gamma = \overline{\Gamma_0(5)^+}$, an arithmetically defined ‘‘moonshine group’’ (see [14] and references therein), we show that $A_M = e^{\ell_{M,0}}$. It is even more surprising that in the case of the group $\Gamma = \overline{\Gamma_0(6)^+}$, that is of the same signature as $\Gamma = \overline{\Gamma_0(5)^+}$, we again have the relation $A_M = (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Consequently, we conclude that the study of the vertical counting function $N_{\text{ver}}(T; (Z_M H_M)')$ contains a term which provides new information associated to Γ which we do not see as being previously detected.

The philosophy behind the Phillips-Sarnak conjecture suggests that the spectral analysis of the Laplacian acting on smooth functions on M should depend on the arithmetic nature of the underlying Fuchsian group Γ . One realization of the spectral analysis of the Laplacian is the location of zeros of Z_M , which includes as a special consideration the function $N_{\text{ver}}(T; Z_M)$, or, equivalently, the function $N_{\text{ver}}(T; Z_M H_M)$. If we are allowed to view the vertical counting function $N_{\text{ver}}(T; (Z_M H_M)')$ as another measure of the spectral analysis of the Laplacian on M , then our Main Theorem shows the existence of refined information, namely A_M with its conditional definition (9), about the uniformizing group Γ . In addition, we found that the value of A_M is different for two different arithmetically defined discrete groups with the same signature. We view this conclusion quite surprising, yet in full support of the Phillips-Sarnak philosophy.

1.8 Computations for the modular group

After the completion of this article, W. Luo brought to our attention the unpublished article [48] from 2008 in which the author undertakes a related study in the case when $\Gamma = \text{PSL}(2, \mathbb{Z})$. There are a number of important differences between the results in the present paper and those in [48], which we now discuss.

In [48], as the title of the article states, the author studies the zeros of the derivative of the zeta function $Z_M(s)/\zeta_{\mathbb{Q}}(2s)$ where $M = \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}$. If we restrict our analysis to the case when $\Gamma = \text{PSL}(2, \mathbb{Z})$, then the function whose derivative we study is $Z_M(s)\zeta_{\mathbb{Q}}(2s-1)/\zeta_{\mathbb{Q}}(2s)$. Since the article [48] studies a different function than in the present article, one would expect that the statements of the main results are different, as, indeed, is the case. More importantly, however, the asymptotic expansions obtained in [48] has an error term of $O(T)$, whereas our error term is $o(T)$,

which is significant since the coefficient of the T term contains the quantity A_M , which we view as a new spectral invariant.

Finally, we note that the article [48] studies the single group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. The approach may extend to other settings when one has explicit knowledge of the scattering matrix; however, we do not see how the approach of [48] would apply for general non-arithmetic surfaces. By contrast, our Main Theorem applies to an arbitrary, co-finite group Γ , and the issue of arithmeticity of Γ plays a role only when one is evaluating the invariant A_M .

1.9 Outline of the paper

This article is organized as follows. In Section 2, we will establish notation and recall necessary results from the literature. The zero-free region for $(Z_M H_M)'$, as stated in part (a) of the Main Theorem, will be proved in Section 3. Various lemmas leading up to the proof of parts (b) and (c) of the main Theorem will be given in Section 4, the proof of parts (b) and (c) will be completed in Section 5, and in Section 6 we will state and prove several corollaries of the Main Theorem. The examples of congruence groups and “moonshine” groups will be given in Section 7. In Section 8, we prove results analogous to our Main Theorem for higher derivatives of $Z_M H_M$. Finally, in Section 9, we will give various concluding remarks. In particular, we will review Weyl’s law for M and show that, under a natural restatement, the constant \mathfrak{g}_1 appears.

2 Background material

2.1 Basic notation

Let $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group of the first kind acting on the upper half plane \mathbb{H} , which we parameterize by $z = x + iy \in \mathbb{C}$ with $y > 0$. Let $M = \Gamma \backslash \mathbb{H}$ be the quotient Riemann surface, which of course may have orbifold singularities if Γ has elliptic elements. The upper half plane \mathbb{H} is equipped with the canonical metric with constant negative curvature equal to -1 , which induces a metric on M whose volume $\mathrm{vol}(M)$ is finite. We will assume standard notions in hyperbolic geometry, referring to [6] for further details.

2.2 Counting functions.

Let F denote either a general Dirichlet series with a critical line; F itself may be the derivative of another general Dirichlet series. We assume that F is normalized to be convergent in the half plane $\mathrm{Re}(s) > 1$ with critical line $\mathrm{Re}(s) = 1/2$. We define the vertical counting function of F as

$$N_{\mathrm{ver}}(T; F) = \sum_{\substack{F(\sigma+it)=0 \\ 0 < t < T, 0 < \sigma < 1}} 1$$

and the horizontal counting function of F as

$$N_{\mathrm{hor}}(T; F) = \sum_{\substack{F(\sigma+it)=0 \\ 0 < t < T, 1/2 < \sigma < 1}} (\sigma - 1/2).$$

Classical results study the vertical and horizontal counting functions when F is a zeta function from an algebraic number field, with more recent attention turned to the setting when F is the derivative of such a zeta function, as discussed in the introduction.

2.3 The Selberg zeta function

Let $\mathcal{H}(\Gamma)$ denote a complete set of representatives of inconjugate, primitive hyperbolic elements of Γ . For each $P \in \mathcal{H}(\Gamma)$, there exists an element $P_0 \in \mathcal{H}(\Gamma)$ such that $P = P_0^n$ for some positive integer n . The element P_0 is called a primitive element of $\mathcal{H}(\Gamma)$. If ℓ_P denotes the length of the geodesic path in the homotopy class determined by P , then the norm of the element P , denoted by $N(P)$ is

equal to $\exp(\ell_P)$. For $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 1$, the Selberg zeta function $Z_M(s)$ is formally defined by the Euler product

$$Z_M(s) = \prod_{n=0}^{\infty} \prod_{P_0 \in \mathcal{H}(\Gamma)} \left(1 - e^{-(s+n)\ell_{P_0}}\right) = \prod_{n=0}^{\infty} \prod_{P_0 \in \mathcal{H}(\Gamma)} \left(1 - N(P_0)^{-(s+n)}\right). \quad (14)$$

The product (14) is defined for $\operatorname{Re}(s) > 1$ and admits a meromorphic continuation to the entire complex plane with the functional equation $Z_M(s)\phi_M(s) = \eta_M(s)Z_M(1-s)$ where

$$\eta_M(s) = \eta_M(1/2) \exp \left(\int_{1/2}^s \frac{\eta'_M(u)}{\eta_M(u)} du \right),$$

and

$$\begin{aligned} \frac{\eta'_M(s)}{\eta_M(s)} &= \operatorname{vol}(M)(s-1/2) \tan(\pi(s-1/2)) - \pi \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{1}{M_R \sin \theta} \frac{\cos(2\theta - \pi)(s-1/2)}{\cos \pi(s-1/2)} \\ &\quad + 2n_1 \log 2 + n_1 \left(\frac{\Gamma'}{\Gamma}(1/2+s) + \frac{\Gamma'}{\Gamma}(3/2-s) \right) \\ &= \frac{\eta'_M}{\eta_M}(1-s). \end{aligned} \quad (15)$$

The set $\{R\}$ denotes the finite set of inconjugate elliptic elements of Γ and $0 < \theta(R) < \pi$ is uniquely determined real number such that elliptic element R is conjugate to the matrix

$$\begin{pmatrix} \cos \theta(R) & -\sin \theta(R) \\ \sin \theta(R) & \cos \theta(R) \end{pmatrix}.$$

We refer the interested reader to [29] for a proof of meromorphic continuation and functional equation of $Z_M(s)$; specifically, see pp. 499-500 in the case where, in the notation of [29], $m = 0$, $r = 1$ and $W = \operatorname{Id}$.

The divisor of the Selberg zeta function is stated as Theorem 5.3, page 498 of [29]. In brief, there are trivial zeros and poles at the negative integers and half-integers, as well as a finite number of poles at $s = 1/2$ and $s = s_n \in (0, 1/2)$, where $s_n(1-s_n) \in (0, 1)$ is a small eigenvalue of the Laplacian. Additionally, there are zeros at points of the form $1/2 + ir_n$, where $1/4 + r_n^2$ is an eigenvalue of the Laplacian and at points in the half-plane $\operatorname{Re}(s) < 1/2$ coming from the non-trivial factor ϕ_M in the functional equation. The function ϕ_M is the determinant of the scattering matrix and is described in section 1.3 above. The reader is referred to page 498 of [29] for additional background material on this matter.

2.4 Additional identities

For any element $P \in \mathcal{H}(\Gamma)$, let $P_0 \in \mathcal{H}(\Gamma)$ be the unique primitive hyperbolic element such that $P = P_0^n$, for some positive integer n . The logarithmic derivative

$$D_M(s) := \frac{Z'_M(s)}{Z_M(s)} \quad (16)$$

of the Selberg zeta function may be expressed, for $\operatorname{Re}(s) > 1$ as the absolutely convergent series

$$D_M(s) = \sum_{P \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s}, \quad (17)$$

where

$$\Lambda(P) := \frac{\log N(P_0)}{1 - N(P)^{-1}};$$

one views $\Lambda(P)$ as the analogue of the classical von Mangoldt Λ function.

Dirichlet series representation of the logarithmic derivative of the function $Z_M H_M$ is given by the following lemma.

Lemma 1 *There exists a constant $\sigma'_0 \geq 1$ such that for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \sigma'_0 + \epsilon > \sigma'_0$, we have that*

$$\frac{(Z_M H_M)'}{(Z_M H_M)}(s) = \sum_{P \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s}. \quad (18)$$

In addition, the series converge absolutely and uniformly on every compact subset of the half plane $\operatorname{Re}(s) > \sigma'_0$.

Proof. The proof follows from the elementary formula

$$(Z_M H_M)'(s) = (Z_M(s) H_M(s)) \left(\frac{Z'_M}{Z_M}(s) + \frac{H'_M}{H_M}(s) \right)$$

together with equations (8) and (17). We may take σ'_0 to be equal to σ_0 , which was defined in section 1.4. ■

Lemma 2 *The derivative of the function $Z_M H_M$ satisfies the functional equation*

$$(Z_M H_M)'(s) = f_M(s) \eta_M(s) K_M^{-1}(s) \tilde{Z}_M(1-s) Z_M(1-s), \quad (19)$$

where

$$f_M(s) := \operatorname{vol}(M)(1/2 - s)(\tan \pi(1/2 - s)) \quad (20)$$

and

$$\tilde{Z}_M(s) := \frac{1}{f_M(s)} \left(\frac{\eta'_M}{\eta_M}(s) - \frac{K'_M}{K_M}(1-s) - \frac{Z'_M}{Z_M}(s) \right). \quad (21)$$

Proof. If one writes the functional equation of the Selberg zeta function as

$$Z_M H_M(s) = \eta_M(s) K_M^{-1}(s) Z_M(1-s), \quad (22)$$

the claimed result then follows from straightforward computations in calculus. ■

Let

$$\psi_M(x) := \sum_{N(P) \leq x} \Lambda(P)$$

denote the prime geodesic counting function. The prime geodesic theorem, which we quote from [29], states the asymptotic growth of $\psi_M(x)$. Specifically, Theorem 3.4 on page 474, with the notation that $W = \operatorname{Id}$, states the asymptotic formula

$$\psi_M(x) = \sum_{k=0}^K \frac{x^{s_k}}{s_k} + O(x^{3/4} \sqrt{\log x}) \quad \text{as } x \rightarrow +\infty.$$

In the prime geodesic theorem, we have the notation that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_K < 1/4$ is the sequence of discrete eigenvalues of the Laplacian Δ_M less than $1/4$, and $s_n = 1/2 + \sqrt{\frac{1}{4} - \lambda_n}$. Since $\lambda_0 = 0$, the leading term in the expansion of $\psi_M(x)$ is x .

2.5 An integral representation for $D_M(s)$

In this section we recall results from [5] on the growth of the logarithmic derivative $D_M(s)$ and its derivatives $D_M^{(k)}(s)$ for $s = 1/2 + \sigma + iT$, as $T \rightarrow \pm\infty$, for $\sigma \in (0, 1/2)$.

The upper bounds on the growth of $D_M^{(k)}$, for $k = 0, 1, 2, \dots$ are deduced inductively from a new integral representation of $D_M(s)$. Since the bounds for the logarithmic derivative represent an

important ingredient in the proof of the Main Theorem, we will recall the new integral representation of (16) obtained in [5]; see Theorem 5.1b). We then briefly explain how to deduce the appropriate bounds for $D_M^{(k)}$ for all $k \geq 0$.

For $n = 0, \dots, K$, so then $\lambda_n < 1/4$, we set $r_n = -i\sqrt{\frac{1}{4} - \lambda_n}$. Set

$$r(t) = \tanh \pi t - 1, \quad \text{and} \quad H(t) = \frac{\Gamma'}{\Gamma}(1+it) + \frac{\Gamma'}{\Gamma}(1-it) - 2 \log t.$$

Let $N_M[0 \leq r_n \leq t]$ be the counting function for the number of non-negative numbers $r_n \leq t$ such that $\frac{1}{4} + r_n^2 = \lambda_n$ is an eigenfunction of the Laplacian. Let

$$R_M(t) = N_M[0 \leq r_n \leq t] - \frac{1}{4\pi} \int_{-t}^t \frac{\phi'_M}{\phi_M} \left(\frac{1}{2} + iu \right) du - \frac{\text{vol}(M)}{4\pi} t^2 + \frac{tn_1}{\pi} (\log 2t - 1),$$

which is the error term in the Weyl's law. From [29], namely Theorems 2.28 and 2.29 on pages 466-468, we have that

$$R_M(t) = O\left(\frac{t}{\log t}\right) \quad \text{and} \quad \int_0^t R_M(u) du = O\left(\frac{t}{\log^2 t}\right). \quad (23)$$

With all this, we quote the following result is from [5].

Theorem 3 *Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1/2$ and write $s = 1/2 + \alpha$. Let $y > 0$ be arbitrary and set $x = e^y$. Then, we have the identity*

$$\begin{aligned} D_M(s) = & \frac{1}{1+x^{2\alpha}} \sum_{P \in \mathcal{H}(\Gamma), N(P) < x} \frac{\Lambda(P)}{N(P)^{\alpha+\frac{1}{2}}} (x^{2\alpha} - N(P)^{2\alpha}) \\ & + \frac{4\alpha x^\alpha}{1+x^{2\alpha}} \left(\sum_{n=0}^K \frac{\cos y r_n}{\alpha^2 + r_n^2} + \frac{R_M(0)}{\alpha^2} + \int_0^\infty \frac{\cos yt dR_M(t)}{\alpha^2 + t^2} - \frac{\text{vol}(M)}{2\pi} \int_0^\infty \frac{t \cdot r(t) \cos yt}{\alpha^2 + t^2} dt \right. \\ & \left. + \frac{n_1}{2\pi} \int_0^\infty \frac{\cos yt H(t) dt}{\alpha^2 + t^2} - \sum_{\substack{\{R\}_\Gamma \\ 0 < \theta(R) < \pi}} \frac{1}{2M_R \sin \theta} \int_0^\infty \frac{\cos yt}{\alpha^2 + t^2} \frac{\cosh 2(\pi - \theta)t}{\cosh 2\pi t} dt - \frac{1}{4} \text{Tr} \left(I - \Phi_M\left(\frac{1}{2}\right) \right) \right) \end{aligned} \quad (24)$$

All integrals in (24) converge uniformly in s on every compact subset of the half-plane $\text{Re}(s) > 1/2$.

The representation (24) itself is of a number-theoretic interest, since it allows us to pass to the limit on the right hand side as $y \rightarrow \infty$, while the left hand side does not depend upon y , and obtain some interesting special values of the function $D_M(s)$; see [3] and [4]. The identity (24) is used in [5] to deduce the following theorem.

Theorem 4 *For $s = 1/2 + \sigma + iT$, $0 < \sigma < 1/2$ and every non-negative integer k , we have the asymptotic bound*

$$D_M^{(k)}(s) = O\left(\min\left\{\frac{|T|}{\sigma^{k+1} \log |T|}, |T|^{1-2\sigma} \log^{k-2\sigma} |T| \cdot \max_{j=0, \dots, k} \left\{ \frac{1}{\sigma^{j+1} \log^{j+1} |T|}, \log \left| \frac{T}{\sigma} \right| \right\} \right\}\right), \quad (25)$$

as $|T| \rightarrow \infty$.

Proof. For the sake of completeness, we will recall main steps of the proof from [5].

The first bound for $k = 0$ is obtained by letting $y \rightarrow 0$ in the representation (24) and showing that we may pass to the limit as $y \rightarrow 0$ inside the integrals in (24). The main obstacle is to prove that

$$\int_0^\infty \frac{dR_M(t)}{(s-1/2)^2 + t^2} = O\left(\frac{1}{\sigma \log |T|}\right) \quad \text{as } |T| \rightarrow \infty,$$

which is done by integrating by parts two times and then using the second bound in (23). The bound for $k \geq 1$ is derived by induction.

The second bound in the case when $k = 0$ is derived by choosing $x \sim T^2 \log^2 |T|$ and estimating each term in (24) separately. There are two main issues.

We use the prime geodesic theorem and integration by parts, for $\alpha = \sigma + iT$, $\frac{1}{2} > \delta \geq \sigma > 0$ in order to arrive at the bound

$$\begin{aligned} & \frac{1}{1+x^{2\alpha}} \sum_{N(P) < x} \frac{\Lambda(P)}{N(P)^{\alpha+\frac{1}{2}}} (x^{2\alpha} - N(P)^{2\alpha}) \\ & \ll \frac{1}{1+(T \log |T|)^{4\alpha}} \int_{\eta}^{T^2 \log^2 |T|} \frac{((T \log |T|)^{4\alpha} - t^{2\alpha})}{t^{\alpha+\frac{1}{2}}} d\psi_M(t) \\ & = O(|T \log |T||^{1-2\sigma}) = O\left(|T|^{1-2\sigma} \log^{-2\sigma} |T| \cdot \log \left| \frac{T}{\sigma} \right| \right), \text{ as } |T| \rightarrow \infty \end{aligned}$$

for the first sum in (24). The lower limit $\eta \in (1, N(P_{00}))$ is arbitrary.

Secondly, from the bound

$$\int_0^\infty \frac{\cos ytdR_M(t)}{\alpha^2 + t^2} = O\left(\max\left\{\frac{1}{\sigma \log |T|}, \log |T/\sigma|\right\}\right), \text{ as } |T| \rightarrow \infty$$

we deduce that

$$\frac{4\alpha x^\alpha}{1+x^{2\alpha}} \int_0^\infty \frac{\cos ytdR_M(t)}{\alpha^2 + t^2} = O\left(T^{1-2\sigma} \log^{-2\sigma} |T| \max\left\{\frac{1}{\sigma \log |T|}, \log \left| \frac{T}{\sigma} \right| \right\}\right),$$

for $\alpha = \sigma + iT$, $\frac{1}{2} > \delta \geq \sigma > 0$, as $|T| \rightarrow \infty$.

Other integrals on the right hand side of (24) may be easily estimated using the approximation of functions under the integral sign. This completes the proof of the theorem in the case when $k = 0$.

The proof in the case $k = 1$ is derived by multiplying the formula (24) by $(1+x^{2\alpha})$ then differentiating the resulting identity, after which one estimates of the integrals using the stated result when $k = 0$. Differentiation under the integral sign is justified due to the uniform convergence of the integral for $1/2 > \delta \geq \sigma \geq \sigma_0 > 0$. The case $k > 1$ is proved analogously by induction. ■

2.6 The completed function Ξ_M

In this section we recall the notation and results from [20]. The notation of [20] is adjusted to our setting; we take $k = 0$, dimension $d = 1$ and $\tau^* = n_1$.

The completed function Ξ_M associated to the Selberg zeta function is defined by

$$\Xi_M(s) = \Xi_I(s) \Xi_{M,\text{hyp}}(s) \Xi_{M,\text{par}}(s) \Xi_{M,\text{ell}}(s)$$

where $\Xi_{M,\text{hyp}}(s) = Z_M(s)$ is the Selberg zeta function and the remaining functions are associated to the identity, parabolic, and elliptic elements in the underlying uniformizing group. The logarithmic derivative of the identity term Ξ_I is given by

$$-\frac{1}{2s-1} \frac{\Xi'_I(s)}{\Xi_I(s)} = \frac{\text{vol}(M)}{2\pi} \frac{\Gamma'(s)}{\Gamma(s)}; \quad (26)$$

see Remark 3.1.3 in [20]. The function $\Xi_{M,\text{ell}}(s)$ is computed in Corollary 2.3.5 of [20]; using Stirling's formula, one can show that

$$\frac{1}{2s-1} \frac{\Xi'_{M,\text{ell}}(s)}{\Xi_{M,\text{ell}}(s)} = O\left(\frac{1}{|t|} \log |t|\right) \quad (27)$$

for any $s = \sigma + it$, $\sigma \leq 1/2$, as $|t| \rightarrow \infty$.

The function $\Xi_{M,\text{par}}(s)$ is described in [20]; see Definition 3.1.4. For our purposes, it suffices to relate $\Xi_{M,\text{par}}(s)$ to the scattering determinant $\phi_M(s)$, so that we obtain an expression for $Z_M H_M(s)$. The following computations derive such an expression for $Z_M H_M(s)$.

Let $\{p_1, \dots, p_{N_0}\}$ denote the set of poles of ϕ_M lying in $(1/2, 1]$, counted with multiplicities; let q_1, \dots, q_{N_1} denote the set of real zeros of ϕ_M larger than $1/2$ and let $\{q_n\}_{n > N_1}$ denote the set of zeros of ϕ_M with positive imaginary parts, counted with multiplicities. In the notation of Definition 3.2.2 from [20], we set $\mathcal{P}_M \equiv 1$ if $n_1 = 0$, otherwise we define

$$\mathcal{P}_M(s) := f_1(s)f_2(s)$$

where

$$f_1(s) := \prod_{n=1}^{N_1} \left(1 + \frac{s-1/2}{q_n-1/2}\right) \exp \left[\frac{1}{2} \left(\frac{s-1/2}{q_n-1/2} \right)^2 \right]$$

and

$$f_2(s) := \prod_{n \geq N_1+1} \left(1 + \frac{s-1/2}{q_n-1/2}\right) \left(1 + \frac{s-1/2}{\overline{q_n}-1/2}\right) \exp \left[\frac{1}{2} \left(\frac{s-1/2}{q_n-1/2} \right)^2 + \frac{1}{2} \left(\frac{s-1/2}{\overline{q_n}-1/2} \right)^2 \right].$$

The infinite product which defines f_2 converges uniformly on compact subsets of \mathbb{C} and defines an entire function of finite order.

Lemma 5 *For all $s \in \mathbb{C}$, the product $(\Xi_M \mathcal{P}_M)(1-s)$ can be expressed as*

$$\begin{aligned} (\Xi_M \mathcal{P}_M)(1-s) &= (Z_M H_M)(s) \cdot \Xi_I(s) \cdot \Xi_{M,\text{ell}}(s) \cdot \frac{\pi^{n_1/2} d(1)}{\phi_M(1/2)} \mathfrak{g}_1^{-s-1} \cdot \\ &\quad \left(s - \frac{1}{2}\right)^{\frac{1}{2} \text{Tr}(I_{n_1} - \Phi_M(\frac{1}{2})) - n_1} \cdot \Gamma(s)^{-n_1} \prod_{m=1}^{N_0} \left(\frac{s-p_m}{1/2-p_m} \right). \end{aligned} \quad (28)$$

Proof. From the functional equation (3.2.4) on p. 123 of [20], we have, for all $s \in \mathbb{C}$

$$(\Xi_M \mathcal{P}_M)(1-s) = (\Xi_M \mathcal{P}_M)(s) \mathfrak{g}_1^{2s-1} \prod_{m=1}^{N_0} \left(\frac{s-p_m}{1-s-p_m} \right) \frac{1}{\phi_M(1/2)} \phi_M(s). \quad (29)$$

On the other hand, by the Corollary 2.4.22 of [20] it is easy to see that

$$(\Xi_{M,\text{par}} \mathcal{P}_M)(s) = (s-1/2)^{\frac{1}{2} \text{Tr}(I_{n_1} - \Phi_M(\frac{1}{2}))} \mathfrak{g}_1^{-s} \left(\frac{1}{\Gamma(s+1/2)} \right)^{n_1} \prod_{m=1}^{N_0} \left(1 + \frac{s-1/2}{p_m-1/2} \right).$$

We now write ϕ_M as

$$\phi_M(s) = \pi^{n_1/2} \mathfrak{g}_1^{-2s} d(1) (s-1/2)^{-n_1} \left(\frac{\Gamma(s+1/2)}{\Gamma(s)} \right)^{n_1} H_M(s).$$

The result follows through direct and straightforward computations involving the the definition of Ξ_M together with (29). ■

2.7 Littlewood's theorem

Several components of the main theorem will be provided using a general theorem from complex analysis due to Littlewood, which we now quote from [59].

Let $f(z)$ be a meromorphic functions which is non-zero and has n poles along the rectangular contour \mathcal{C} which is bounded by the lines $x = x_1$, $x = x_2$, $y = y_1$ and $y = y_2$. Let $F(z) = \log f(z)$ be

the logarithm of $f(z)$ defined by analytic continuation along \mathcal{C} , and $N(x'; f, \mathcal{C})$ denote the number of zeros of f minus the number of poles of f in the sub-region of \mathcal{C} where $x > x'$. Then

$$\int_{\mathcal{C}} F(z) dz = -2\pi i \int_{x_1}^{x_2} N(x; f, \mathcal{C}) dx. \quad (30)$$

We refer the reader to [59] for an elementary proof of (30).

2.8 On generalized Backlund equivalent for the Lindelöf hypothesis

An important ingredient in the proof of the Main Theorem is a bound on the growth of the function $Z_M H_M$ on the critical line $\operatorname{Re}(s) = 1/2$. We obtain the bound using a slight modification of Proposition 2 from [24], which we now state.

Proposition 6 *Let $f(s)$ be a meromorphic function for all $s \in \mathbb{C}$ which is holomorphic in the region for $|\operatorname{Im}(s)| \geq t_0 > 0$, for some fixed t_0 . Let $P(t) : \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing function such that $P(t) \geq 2$. Let $N(\sigma, f, T)$ denote the number of zeros ρ of f in the region $\operatorname{Re}(\rho) > \sigma$; $0 \leq \operatorname{Im}(\rho) \leq T$.*

Assume there exist constants $\sigma_0 > 1/2$ and $\omega > 0$ such that for $\sigma_0 - \omega \leq \operatorname{Re}(s) \leq \sigma_0 + \omega$ we have

$$|f(s)| \geq c > 0 \quad \text{and} \quad (f'/f)(s) = o(P(t)) \quad \text{as } t = \operatorname{Im}(s) \rightarrow \infty.$$

Furthermore, assume that $|f(s)| > 0$ for $\operatorname{Re}(s) \geq \sigma_0 + \omega$ and that for some fixed number $D > 0$ we have

$$f(s) = O\left((P(t))^D\right) \quad \text{as } t = \operatorname{Im}(s) \rightarrow \infty, \text{ uniformly for } \operatorname{Re}(s) \geq 2 - 3\sigma_0.$$

Then if the estimate $N(\sigma, f, T+1) - N(\sigma, f, T) = o(P(T))$ holds for all $\sigma \geq 1/2$ as $T \rightarrow \infty$, then $f(1/2 + it) = O_\epsilon((P(t))^\epsilon)$ as $t \rightarrow \infty$.

There are slight differences between our statement above and Proposition 2 from [24]. Firstly, we assume the function f depends on a single complex variable s , not necessarily a member of a family of meromorphic functions. Secondly, the author of [24] assumes that f has a finite number of poles which lie in a compact set, with the goal of applying the result to the Selberg zeta function associated to a compact hyperbolic Riemann surface. A review of the proof of Proposition 2 from [24] reveals that the argument is based on Landau's theorem (see Lemma 8 of [24]) and Hadamard's three circles theorem (see Lemma 9 of [24]). These classical results are applied to the function $f(s)$ in the neighborhood $|s - s_0| \leq 2(\sigma_0 - 1/2 - \delta)$ of the point $s_0 = \sigma_0 + iT$ for sufficiently large T . The proof given in [24] carries through without any changes whatsoever under the assumptions we state above.

We refer the reader to [24] for the proof and various interesting generalizations of Proposition 6.

3 Zeros in a half plane $\operatorname{Re}(s) < 1/2$

In this section we will prove part (a) of the Main Theorem. In fact, we will prove more than stated, since our analysis will yield regions where each of the functions $\operatorname{Re}((Z_M H_M)')$ and $\operatorname{Im}((Z_M H_M)')$ are non-vanishing.

Proposition 7 *a) For $\sigma < 1/2$, there exists $t_0 > 0$, which may depend on σ , such that*

$$\operatorname{Re}((Z_M H_M)'(\sigma + it)) \neq 0 \quad \text{for all } t \text{ such that } |t| > t_0.$$

b) For every constant $C > 0$ and arbitrary $-C < \sigma'_0 < 1/2$ there are at most finitely many zeros of $(Z_M H_M)'(s)$ inside the strip $-C \leq \operatorname{Re}(s) \leq \sigma'_0$

Proof. We first present the proof of part (a). By taking the logarithmic derivative of the functional equation (22) we get for $s = \sigma + it$ with $\sigma < 1/2$, the equation

$$\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) = \frac{\eta'_M}{\eta_M}(\sigma + it) - \frac{K'_M}{K_M}(\sigma + it) - \frac{Z'_M}{Z_M}(1 - \sigma - it). \quad (31)$$

From the definition (15) of η'_M/η_M and K_M , one can use Stirling's formula, together with the bound $0 < \theta < \pi$, to show that

$$\operatorname{Re} \left(\frac{\eta'_M}{\eta_M}(\sigma + it) \right) = -\operatorname{vol}(M)t + O(\log |t|) \quad \text{and} \quad \frac{K'_M}{K_M}(\sigma + it) = O(\log |t|),$$

for $\sigma < 1/2$ and as $|t| \rightarrow \infty$. Therefore,

$$\operatorname{Re} \left(-\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \right) = \operatorname{vol}(M)t + O(\log |t|) + \operatorname{Re} \left(\frac{Z'_M}{Z_M}(1 - \sigma - it) \right).$$

Replacing σ by $1/2 - \sigma$ in (25) we get, for $\sigma < 1/2$,

$$\operatorname{Re} \left(\frac{Z'_M}{Z_M}(1 - \sigma - it) \right) = O \left(\frac{|t|}{(1/2 - \sigma) \log |t|} \right) \quad \text{as } |t| \rightarrow \infty,$$

so then

$$\operatorname{Re} \left(-\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \right) = \operatorname{vol}(M)t + O \left(\frac{|t|}{(1/2 - \sigma) \log |t|} \right),$$

for $\sigma < 1/2$ as $t \rightarrow \pm\infty$. Therefore, there exists $t_0 > 0$ such that

$$\operatorname{Re} \left(\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \right) \neq 0 \quad \text{for all } s = \sigma + it, \text{ with } |t| > t_0.$$

On the other hand, the non-trivial zeros of the function $Z_M H_M$ are either non-trivial zeros $\rho = \frac{1}{2} \pm ir_n$ of Z_M or zeros ρ of ϕ_M . All except finitely many zeros of ϕ_M have real part bigger than $1/2$; therefore, $Z_M H_M(\sigma + it) \neq 0$ for $\sigma < 1/2$ and $t > t_0$. Therefore, we conclude that $\operatorname{Re}((Z_M H_M)'(\sigma + it)) \neq 0$ for all $t > t_0$. With all this, the proof of part a) is complete.

To prove part (b), we employ Lemma 2. Recall the function $\tilde{Z}_M(s)$ which is defined in (21). Let us write $\tilde{Z}_M(s) = 1 + Z_{M,1}(s)$. Then

$$\begin{aligned} -f_M(s)Z_{M,1}(s) &= \pi \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{1}{M_R \sin \theta} \frac{\cos(2\theta - \pi)(s - 1/2)}{\cos \pi(s - 1/2)} - 2n_1 \log 2 \\ &\quad - n_1 \left(\frac{\Gamma'}{\Gamma}(1/2 + s) + \frac{\Gamma'}{\Gamma}(3/2 - s) - \frac{\Gamma'}{\Gamma}(1/2 - s) + \frac{\Gamma'}{\Gamma}(1 - s) \right) + \frac{Z'_M}{Z_M}(s), \end{aligned} \quad (32)$$

where f_M is defined in (20).

As in the proof of part (a), we can use Stirling's formula and (25) to arrive at the bound

$$Z_{M,1}(\sigma_1 + it) = O \left(\frac{(|t| \log |t|)^{2-2\sigma_1}}{(\sigma_1 - 1/2)|t|} \right) \quad \text{as } |t| \rightarrow \infty.$$

for $\sigma_1 > 1/2$ and $(\sigma_1 + it) \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} B_n$ where B_n are small circles of fixed radius centered at integers. In particular, for $\sigma_1 > 1/2$ and $(\sigma_1 + it) \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} B_n$, function $Z_{M,1}(\sigma_1 + iT)$ is uniformly bounded in T . Therefore, $\tilde{Z}_M(1-s) = 1 + o(1)$, as $\text{Im}(s) \rightarrow \pm\infty$ in the strip $-C \leq \text{Re}(s) \leq \sigma'_0 < 1/2$, hence $\tilde{Z}_M(1-s)$ has finitely many zeros in this strip.

Since $Z_M(s)$ has only finitely many zeros for $\text{Re}(s) > 1/2$, the function $\tilde{Z}_M(1-s)Z_M(1-s)$ will have finitely many zeros in the strip $-C \leq \text{Re}(s) \leq \sigma'_0 < 1/2$. Equation (19) then implies that the set of zeros of $(Z_M H_M)'(s)$ in the strip $-C \leq \text{Re}(s) \leq \sigma'_0 < 1/2$ is finite since the factor $\Phi_M(s) := f_M(s)\eta_M(s)K_M^{-1}(s)$ of the functional equation (19) also has at most finitely many zeros in this strip. ■

We note that the zeros of $(Z_M H_M)'(s)$ which arise from zeros of Φ_M can be viewed as trivial zeros. The trivial zeros of $(Z_M H_M)'(s)$ are in the region $\text{Re}(s) < 1/2$ and arise at all negative integers.

The above method of examining zeros of the function $(Z_M H_M)'$ has the critical line as its limitation, since the integral representation (24) and the bounds for the logarithmic derivative (25) hold true only in the half plane $\text{Re}(s) > 1/2$. In order to derive results valid on the critical line we need a representation on the critical line. Such a representation exists for the complete zeta function $\Xi_M(s)$ recalled in section 2.7.

Proposition 8 *There exists a number $t_0 > 0$ such that the following statements hold:*

- a) $\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \neq 0$ for all $\sigma < 1/2$ and all $|t| > t_0$;
- b) $\frac{(Z_M H_M)'}{(Z_M H_M)}(1/2 + it) \neq 0$ for all $|t| > t_0$, $t \neq r_n$ for all $n \geq 0$.

Proof. Let $\sigma < 0$. By Proposition 7, there exists a constant $t'_0 > 0$ such that for all $\sigma < 0$ and all $|t| > t'_0$ we have $\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \neq 0$. Therefore, it is enough to prove the statement when $0 \leq \sigma < 1/2$. Without loss of generality, we will assume that $t > 0$.

By taking the logarithmic derivatives of the both sides of the equation (28), we get

$$\begin{aligned} \frac{1}{2s-1} \frac{(Z_M H_M)'}{(Z_M H_M)}(s) &= -\frac{1}{2s-1} \frac{\Xi'_I(s)}{\Xi_I(s)} - \frac{1}{2s-1} \frac{\Xi'_{M,\text{ell}}(s)}{\Xi_{M,\text{ell}}(s)} + \frac{\log \mathfrak{g}_1}{2s-1} + \frac{n_1}{2s-1} \frac{\Gamma'}{\Gamma}(s) \\ &+ \frac{n_1 - \frac{1}{2} \text{Tr}(I_{n_1} - \Phi(\frac{1}{2}))}{2(s-1/2)^2} - \frac{1}{2s-1} \sum_{m=1}^{N_0} \frac{1}{s-p_m} - \frac{1}{2s-1} \frac{(\Xi_M \mathcal{P}_M)'}{(\Xi_M \mathcal{P}_M)}(1-s), \end{aligned} \quad (33)$$

for all $s \in \mathbb{C}$ different from zeros and poles of Z_M and ϕ_M . Formula (3.4.1) on page 146 of [20] gives the formula

$$\begin{aligned} (\Xi_M \mathcal{P}_M)(s) &= e^{Q(s)} (s-1/2)^{2d_{1/4}} \prod_{n \geq 0, r_n \neq 0} \left(1 + \frac{(s-1/2)^2}{r_n^2}\right) \exp\left(-\frac{(s-1/2)^2}{r_n^2}\right) \\ &\cdot \prod_{n=1}^{N_1} \left(1 - \frac{s-1/2}{\eta_n}\right) \exp\left(-\frac{s-1/2}{\eta_n} + \frac{(s-1/2)^2}{2\eta_n^2}\right) \\ &\prod_{n \geq N_1+1} \left(1 + \frac{s-1/2}{\eta_n + i\gamma_n}\right) \left(1 + \frac{s-1/2}{\eta_n - i\gamma_n}\right) \exp\left(-\frac{2\eta_n(s-\frac{1}{2})}{\eta_n^2 + \gamma_n^2} + \left(s - \frac{1}{2}\right)^2 \frac{\eta_n^2 - \gamma_n^2}{(\eta_n^2 + \gamma_n^2)^2}\right), \end{aligned}$$

for all $s \in \mathbb{C}$ and where the notation is as follows: $\eta_n := \text{Re}(q_n)$; $\gamma_n := \text{Im}(q_n)$, $d_{1/4}$ is the multiplicity of $\lambda = 1/4$ as an eigenvalue; and $Q(s) = a_2(s-1/2)^2 + a_1(s-1/2) + a_0$ for some constants a_i ,

$i = 0, 1, 2$ computed in [20]. The constants a_1 and a_2 are defined by formulas (3.4.8) and (3.4.9) on page 153 of [20]. For our purposes it is important to know that a_1 and a_2 are real.

We now compute the logarithmic derivative of $(\Xi_M \mathcal{P}_M)(s)$ and substitute the expression into (33). After some elementary calculations, having in mind (26) and (27) we end up with

$$\begin{aligned} \frac{1}{2s-1} \frac{(Z_M H_M)'}{(Z_M H_M)}(s) &= \frac{\text{vol}(M)}{2\pi} \frac{\Gamma'}{\Gamma}(s) + O\left(\frac{\log|t|}{|t|}\right) + \frac{\log \mathfrak{g}_1}{2s-1} + \frac{n_1}{2s-1} \frac{\Gamma'}{\Gamma}(s) \\ &+ \frac{n_1 - \frac{1}{2} \text{Tr}(I_{n_1} - \Phi(\frac{1}{2}))}{2(s-1/2)^2} - \frac{1}{2s-1} \sum_{m=1}^{N_0} \frac{1}{s-p_m} + a_2 - \frac{a_1}{2(s-\frac{1}{2})} + \frac{d_{1/4}}{(s-\frac{1}{2})^2} \\ &+ \sum_{n \geq 0, r_n \neq 0} \left(\frac{1}{(s-\frac{1}{2})^2 + r_n^2} - \frac{1}{r_n^2} \right) + \frac{1}{2} \sum_{n=1}^{N_1} \left(\frac{1}{\eta_n^2} - \frac{1}{\eta_n(\eta_n - s + 1/2)} \right) \\ &+ \sum_{n \geq N_1+1} \left[\frac{\eta_n^2 - \gamma_n^2}{(\eta_n^2 + \gamma_n^2)^2} + \frac{\gamma_n^2 - \eta_n^2 + \eta_n(s-1/2)}{((\eta_n - s + 1/2)^2 + \gamma_n^2)(\eta_n^2 + \gamma_n^2)} \right]. \end{aligned} \quad (34)$$

Since $(\Gamma'/\Gamma)(\sigma + it) = O(\log|\sigma + it|)$, as $t \rightarrow \infty$, from (34) we get

$$\begin{aligned} \frac{1}{2s-1} \frac{(Z_M H_M)'}{(Z_M H_M)}(s) &= \frac{\text{vol}(M)}{2\pi} \frac{\Gamma'}{\Gamma}(s) + \sum_{n \geq 0, r_n \neq 0} \left(\frac{1}{(s-\frac{1}{2})^2 + r_n^2} - \frac{1}{r_n^2} \right) \\ &+ a_2 + \sum_{n \geq N_1+1} \left[\frac{\eta_n^2 - \gamma_n^2}{(\eta_n^2 + \gamma_n^2)^2} + \frac{\gamma_n^2 - \eta_n^2 + \eta_n(s-1/2)}{((\eta_n - s + 1/2)^2 + \gamma_n^2)(\eta_n^2 + \gamma_n^2)} \right] \\ &+ \frac{1}{2} \sum_{n=1}^{N_1} \left(\frac{1}{\eta_n^2} - \frac{1}{\eta_n(\eta_n - s + 1/2)} \right) + O\left(\frac{\log|t|}{|t|}\right), \end{aligned} \quad (35)$$

as $t = \text{Im}(s) \rightarrow \infty$. We now set $s = \sigma + it$ with $t > 0$ and $0 \leq \sigma < 1/2$. By computing the imaginary parts of both sides (35) we get

$$\begin{aligned} \text{Im} \left(\frac{1}{2\sigma-1+2it} \frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \right) &= \frac{\text{vol}(M)}{2\pi} \cdot \left[\frac{t}{\sigma^2 + t^2} + \sum_{n=1}^{\infty} \frac{t}{(n+\sigma)^2 + t^2} \right] \\ &+ \sum_{n \geq 0, r_n \neq 0} \frac{t(1/2 - \sigma)}{((\sigma - 1/2)^2 - t^2 + r_n^2)^2 + 4t^2(\sigma - 1/2)^2} + O\left(\frac{1}{t}\right) + O\left(\frac{\log t}{t}\right) \\ &+ \sum_{n \geq N_1+1} \frac{t\eta_n(3\gamma_n^2 - t^2) - t\eta_n^3 + t(1/2 - \sigma)(2\gamma_n^2 - 2\eta_n^2 - \eta_n(1/2 - \sigma))}{\left[((\eta_n - \sigma + 1/2)^2 + \gamma_n^2 - t^2)^2 + 4t^2(\eta_n + 1/2 - \sigma)^2 \right] (\eta_n^2 + \gamma_n^2)}. \end{aligned} \quad (36)$$

Since $0 \leq \sigma < 1/2$ we have that $(n+\sigma)^2 < (n+1/2)^2$, for all $n \geq 0$. Therefore

$$\frac{t}{\sigma^2 + t^2} + \sum_{n=1}^{\infty} \frac{t}{(n+\sigma)^2 + t^2} > \frac{t}{1/4 + t^2} + \sum_{n=1}^{\infty} \frac{t}{(n+1/2)^2 + t^2} = \frac{\pi}{2} \tanh(\pi t).$$

Furthermore, since $0 < \eta_n < c$, for some positive constant c and all $n \geq 1$ and $\gamma_n \rightarrow \infty$, as $n \rightarrow \infty$, by the choice of σ we have that

$$(1/2 - \sigma)(2\gamma_n^2 - 2\eta_n^2 - \eta_n(1/2 - \sigma)) \geq 2\gamma_n^2 - 2\eta_n^2 - \eta_n \geq 0$$

for all but finitely many $n \geq (N_1 + 1)$. Let $n_1 \geq (N_1 + 1)$ be an integer such that $2\gamma_n^2 - 2\eta_n^2 - \eta_n \geq 0$ for all $n \geq n_1$. For simplicity, we will introduce the notation

$$D(n, \sigma, t) = \left[((\eta_n - \sigma + 1/2)^2 + \gamma_n^2 - t^2)^2 + 4t^2(\eta_n + 1/2 - \sigma)^2 \right].$$

From (36), we conclude the existence a constant $C_1 > 0$ and a positive number $t_0 > t'_0$ such that for all $t > t_0$,

$$\begin{aligned} \operatorname{Im} \left(\frac{1}{2\sigma - 1 + 2it} \frac{(Z_M H_M)'}{(Z_M H_M)} (\sigma + it) \right) &\geq \frac{\operatorname{vol}(M)}{4} \tanh(\pi t) - C_1 \frac{\log t}{t} \\ &+ \sum_{|\gamma_n| < t/\sqrt{3}} \frac{t\eta_n(3\gamma_n^2 - t^2)}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)} + \sum_{N_1+1 \leq n \leq n_1} \frac{(1/2 - \sigma)t(2\gamma_n^2 - 2\eta_n^2 - \eta_n)}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)} \\ &+ \sum_{n \geq N_1+1} \frac{-t\eta_n^3}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)}. \end{aligned} \quad (37)$$

Observe that each term in each summand in (37) is negative. We investigate separately each of the three sums on the right hand side of (37).

Since $(\eta_n - \sigma + 1/2)^2$ is bounded by some constant, by enlarging t_0 if necessary, we get that $D(n, \sigma, t) \geq t^4/4$ for all n such that $|\gamma_n| < t/\sqrt{3}$ and all $t > t_0$. Therefore,

$$\begin{aligned} 0 \leq \sum_{|\gamma_n| < t/\sqrt{3}} \frac{t\eta_n(t^2 - 3\gamma_n^2)}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)} &\leq \sum_{|\gamma_n| < t/\sqrt{3}} \frac{t^3\eta_n}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)} \\ &\leq \frac{2}{t} \sum_{|\gamma_n| < t/\sqrt{3}} \frac{2\eta_n}{(\eta_n^2 + \gamma_n^2)} = O(1/t), \end{aligned}$$

as $t \rightarrow \infty$ because the series

$$\sum_{n \geq N_1+1} \frac{2\eta_n}{(\eta_n^2 + \gamma_n^2)}$$

converges; see Corollary 2.4.17 of [20].

For the second and the third sum in (37) we use the elementary inequality

$$D(n, \sigma, t) \geq 4t^2(\eta_n + 1/2 - \sigma)^2 \geq 4t^2\eta_n^2$$

to deduce that second sum is $O(1/t)$ as $t \rightarrow \infty$. For the third sum, we have that

$$\sum_{n \geq N_1+1} \frac{-t\eta_n^3}{D(n, \sigma, t)(\eta_n^2 + \gamma_n^2)} \geq -\frac{1}{8t} \sum_{n \geq N_1+1} \frac{2\eta_n}{(\eta_n^2 + \gamma_n^2)} = O(1/t)$$

as $t \rightarrow \infty$, again by Corollary 2.4.17 of [20].

The three sums in (37) are $O(1/t)$, as $t \rightarrow \infty$. Hence, there exists a constant $C_2 > 0$, such that for all $t > t_0$, $t \neq r_n$ and $0 \leq \sigma \leq 1/2$ one has

$$\operatorname{Im} \left(\frac{1}{2\sigma - 1 + 2it} \frac{(Z_M H_M)'}{(Z_M H_M)} (\sigma + it) \right) \geq \frac{\operatorname{vol}(M)}{4} \cdot \tanh \pi t - \frac{C_1 \log t + C_2}{t}.$$

Since $\tanh \pi t = 1 + O(e^{-\pi t})$, as $t \rightarrow \infty$ we conclude that

$$\operatorname{Im} \left(\frac{(Z_M H_M)'}{(Z_M H_M)} (\sigma + it) \right) \neq 0 \quad \text{for } t > t_0.$$

With all this, the proof of part (a) is complete.

We now prove part (b). We put $s = 1/2 + it$ for $t > 0$ with $t \neq r_n$ in (36) to get

$$\operatorname{Im} \left(\frac{1}{2it} \frac{(Z_M H_M)'}{(Z_M H_M)} \left(\frac{1}{2} + it \right) \right) = \frac{\operatorname{vol}(M)}{2\pi} \cdot \frac{\pi}{2} \tanh(\pi t) + O \left(\frac{\log t}{t} \right) + \sum_{n \geq N_1+1} \frac{t\eta_n(3\gamma_n^2 - t^2) - t\eta_n^3}{D(n, 1/2, t)(\eta_n^2 + \gamma_n^2)}.$$

Analogously as in the proof of part a) we deduce that there exist a constant $t_1 > 0$ such that

$$\operatorname{Im} \left(\frac{1}{2it} \frac{(Z_M H_M)'}{(Z_M H_M)} \left(\frac{1}{2} + it \right) \right) \geq \frac{\operatorname{vol}(M)}{4} \cdot \tanh \pi t - \frac{C_1 \log t + C_2}{t}.$$

for all $t > t_1$, $t \neq r_n$ and the proof of part (a) is complete. ■

We can now give a proof of part (a) of the Main Theorem.

The function $(Z_M H_M)(s)$ has finitely many non-trivial zeros in the region $\operatorname{Re}(s) < 1/2$. Combining this statement with Proposition 8(a) immediately implies there existence a of constant t_0 such that $(Z_M H_M)'(\sigma + it) \neq 0$ for $\sigma < 1/2$ and $|t| > t_0$.

Proposition 8b) yields that $\frac{(Z_M H_M)'}{(Z_M H_M)}(1/2 + it) \neq 0$ for $|t| > t_0$, $t \neq r_n$ for all $n \geq 1$. Therefore, the only zeros of $(Z_M H_M)'$ on the line $\operatorname{Re} s = 1/2$, with at most a finite number of exceptions, are multiple zeros of $(Z_M H_M)$, or, equivalently, multiple zeros of Z_M .

4 Preliminary lemmas

In this section we will prove some preliminary results needed in the proof of the Main Theorem. Let quantity A_M , resp. a_M , be defined by (9) and (10), resp. (11). Let $P_{00} \in \mathcal{H}(\Gamma)$ denote the primitive hyperbolic element of Γ with the property that $N(P_{00}) = e^{\ell_{M,0}}$; or equivalently, with the property that $N(P_{00}) = \min\{N(P) : P \in \mathcal{H}(\Gamma)\}$. In the case when $A_M = e^{\ell_{M,0}}$ we may write a_M in terms of the norm of P_{00} , namely $a_M = m_{M,0}\Lambda(P_{00})$.

The proof of parts b) and c) of our Main Theorem relies essentially on application of Littlewood's theorem to the function $X_M(s)$ defined for all complex s by

$$X_M(s) := \frac{A_M^s}{a_M} (Z_M H_M)'(s). \quad (38)$$

Lemma 9 *There exist $\sigma_1 > 1$ and a constant $0 < c_\Gamma < 1$ such that for $\sigma = \operatorname{Re}(s) \geq \sigma_1$, we have the asymptotic formula*

$$X_M(s) = 1 + O(c_\Gamma^\sigma) \neq 0,$$

as $\sigma \rightarrow +\infty$.

Proof. From the Euler product definition (14) of Z_M , we have that

$$Z_M(s) = 1 + O \left(\frac{1}{N(P_{00})^{\operatorname{Re}(s)}} \right),$$

as $\operatorname{Re}(s) \rightarrow \infty$. Analogously, from (7), we have that

$$H_M(s) = 1 + O \left(\frac{1}{r_2^{2\operatorname{Re}(s)}} \right)$$

as $\operatorname{Re}(s) \rightarrow +\infty$. Furthermore,

$$\sum_{\{P\} \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s} = \frac{m_{M,0}\Lambda(P_{00})}{N(P_{00})^s} \left(1 + O \left(\frac{1}{C_{\Gamma,1}^{\operatorname{Re}(s)}} \right) \right) + \frac{b((\mathfrak{g}_2/\mathfrak{g}_1)^2)}{(\mathfrak{g}_2/\mathfrak{g}_1)^{2s}} \left(1 + O \left(\frac{1}{C_{\Gamma,2}^{\operatorname{Re}(s)}} \right) \right)$$

as $\text{Re}(s) \rightarrow +\infty$, where

$$C_{\Gamma,1} := \min\{N(P) : N(P) \neq N(P_{00})\}/N(P_{00}) > 1$$

and

$$C_{\Gamma,2} := \min\{q_i : q_i \neq (\mathfrak{g}_2/\mathfrak{g}_1)^2\}/(\mathfrak{g}_2/\mathfrak{g}_1)^2 > 1$$

are constants depending on the underlying group Γ . By the definition of A_M and a_M , we immediately deduce that

$$\sum_{\{P\} \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s} = \frac{a_M}{A_M^s} \left(1 + O\left(\frac{1}{A_{\Gamma,1}^{\text{Re}(s)}}\right) \right), \text{ as } \text{Re}(s) \rightarrow +\infty,$$

for some constant $A_{\Gamma,1} > 1$. We now use Lemma 1, where we multiply the formula (18) by $(Z_M H_M)(s)$, to get

$$(Z_M H_M)'(s) = \frac{a_M}{A_M^s} \left[1 + O\left(\frac{1}{N(P_{00})^{\text{Re}(s)}}\right) \right] \left[1 + O\left(\frac{1}{r_1^{2\text{Re}(s)}}\right) \right] \left[1 + O\left(\frac{1}{A_{\Gamma,1}^{\text{Re}(s)}}\right) \right],$$

as $\text{Re}(s) \rightarrow +\infty$. This completes the proof. ■

The following lemma provides the bound for the growth of the function $Z_{M,1}(s)$; recall that $Z_{M,1}(s)$ is defined by (32).

Lemma 10 *Let $0 < a < 1/2$ be an arbitrary real number and let $\sigma \geq 1 - a$. Then*

$$\log |1 + Z_{M,1}(\sigma \pm iT)| = O(|Z_{M,1}(\sigma \pm iT)|) = O(T^{2a-1} \log^{2a} T), \quad (39)$$

as $T \rightarrow \infty$.

Proof. From the bound (25) with $k = 0$ and $\sigma \geq 1 - a$, we get

$$\frac{Z'_M}{Z_M}(\sigma \pm iT) = O\left((T \log T)^{1-2(\sigma-1/2)}\right) = O((T \log T)^{2a}), \text{ as } T \rightarrow \infty,$$

where the implied constant depends only upon M and a . We then can argue in the same manner as in the proof of Proposition 7b. Namely, we apply Stirling's formula and the above estimate, we get, for $s = \sigma \pm iT$ and $T \geq 1$, the estimate

$$\left| \frac{1}{f_M(s)} \left(\frac{\eta'_M}{\eta_M}(s) - \frac{K'_M}{K_M}(1-s) - \frac{Z'_M}{Z_M}(s) \right) - 1 \right| = O\left(\frac{\log T}{T} + \frac{(T \log T)^{2a}}{T}\right),$$

as $T \rightarrow \infty$. This implies the bound (39) as claimed. ■

The following lemma is a Phragmén-Lindelöf type bound for $(Z_M H_M)$. The bound will be used to derive a similar bound for $(Z_M H_M)'$ using the Cauchy formula.

Lemma 11 *Let $\sigma_2 \geq 1$ be a fixed real number, such that $-\sigma_2$ is not a pole of $(Z_M H_M)$. Then, for an arbitrary $\delta > 0$*

a)

$$(Z_M H_M)(\sigma + it) = O_{\Gamma} \left(\exp \left(\frac{1}{2} + \sigma_2 + \delta \right) \text{vol}(M)t \right),$$

b)

$$Z_M(\sigma + it) = O_{\Gamma} \left(\exp \left(\frac{1}{2} + \sigma_2 \right) \text{vol}(M)t \right)$$

for $t \geq 1$, uniformly in $\sigma \leq -\sigma_2$.

Proof. To prove part (a), we will apply the Phragmen-Lindelöf theorem (e.g. Chapter 5.61. of [59]) to the function

$$F(s) = (Z_M H_M)(s) \exp \left[\text{vol}(M) \left(\frac{1}{2} + \sigma_2 + \delta \right) is \right]$$

which is an entire function of finite order at most two in the sector $D := \{\pi/4 \leq \arg(s + \sigma_2) \leq \pi/2\}$. Obviously, $(Z_M H_M)(s) = O(1)$ along the line $\arg(s + \sigma_2) = \pi/4$, since $(Z_M H_M)(\sigma + it) = O(1)$, for $\sigma > \sigma_1$ and $t \geq 1$; see the proof of Lemma 9. Therefore,

$$|F(s)| = O(1) \quad \text{along the line } \arg(s + \sigma_2) = \pi/4.$$

To determine the behavior of the function $F(s)$ along the vertical line $\arg(s + \sigma_2) = \pi/2$, i.e. for $s = -\sigma_2 + it$, $t \geq 0$, we use the functional equation (22) to get

$$\begin{aligned} |F(-\sigma_2 + it)| &= \exp\left(-\left(\frac{1}{2} + \sigma_2 + \delta\right) \text{vol}(M)t\right) |\eta(-\sigma_2 + it)| |K_M^{-1}(-\sigma_2 + it)| |Z_M(1 + \sigma_2 - it)| \\ &= \exp\left(-\left(\frac{1}{2} + \sigma_2 + \delta\right) \text{vol}(M)t\right) |\eta_M(-\sigma_2 + it)| |K_M^{-1}(-\sigma_2 + it)| \cdot O(1), \end{aligned} \quad (40)$$

since $1 + \sigma_2 \geq 2$. It remains to estimate $|\eta_M(-\sigma_2 + it)|$. Firstly, by (15)

$$\begin{aligned} |\eta_M(-\sigma_2 + it)| &= \exp \left(\text{Re} \int_{1/2}^{-\sigma_2 + it} \frac{\eta'_M(u)}{\eta_M(u)} du \right) = \exp \left(\text{Re} \int_{1/2}^{-\sigma_2 + it} \text{vol}(M)(u) \tan \pi u \cdot du \right) \cdot \\ &\cdot \exp(2n_1 \log 2 \cdot (-\sigma_2 - 1/2)) \cdot \exp \left(n_1 \text{Re} \left(\int_{1/2}^{-\sigma_2 + it} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + u \right) + \frac{\Gamma'}{\Gamma} \left(\frac{3}{2} - u \right) \right) du \right) \right) \\ &\cdot \exp \left(\text{Re} \left(- \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{\pi}{M_R \sin \theta} \int_{1/2}^{-\sigma_2 + it} \frac{\cos(2\theta - \pi)(s - 1/2)}{\cos \pi(s - 1/2)} ds \right) \right). \end{aligned} \quad (41)$$

Formula (4.4) from page 76, of [28], following the notation from the previous page, states the estimate

$$\begin{aligned} &\exp \left(\text{Re} \int_0^{-\sigma_2 - 1/2 + it} \text{vol}(M)(u - 1/2) \tan \pi(u - 1/2) du \right) \\ &= \exp \left(\text{Re} \left(\frac{i}{2} \text{vol}(M) (it - \sigma_2 - 1/2)^2 \right) + O(1) \right) = \exp \left(\text{vol}(M) \left(\frac{1}{2} + \sigma_2 \right) t + O(1) \right) \end{aligned} \quad (42)$$

as $t \rightarrow \infty$. By Stirling's formula,

$$\text{Re} \left(\int_{1/2}^{-\sigma_2 + it} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + u \right) + \frac{\Gamma'}{\Gamma} \left(\frac{3}{2} - u \right) \right) du \right) = O(\log(t)) \quad \text{as } t \rightarrow \infty. \quad (43)$$

The contribution of the elliptic elements is given by

$$\operatorname{Re} \int_{1/2}^{-\sigma_2+it} \frac{\cos(2\theta-\pi)(s-1/2)}{\cos \pi(s-1/2)} ds = \operatorname{Re} \left(\int_{1/2}^{-\sigma_2} \frac{\cos(2\theta-\pi)(\sigma-1/2+it)}{\cos \pi(\sigma-1/2+it)} d\sigma \right).$$

Trivially, one has that

$$\left| \frac{\cos(2\theta-\pi)(\sigma-1/2+it)}{\cos \pi(\sigma-1/2+it)} \right| = O \left(\frac{\exp(|2\theta-\pi|t)}{\exp(\pi t)} \right) \quad \text{as } t \rightarrow +\infty.$$

Since $0 < \theta < \pi$, $|2\theta-\pi|t - \pi t < 0$, hence

$$\operatorname{Re} \left(- \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{\pi}{M_R \sin \theta} \int_{1/2}^{-\sigma_2+it} \frac{\cos(2\theta-\pi)(s-1/2)}{\cos \pi(s-1/2)} ds \right) = O(1) \quad \text{as } t \rightarrow +\infty. \quad (44)$$

Substituting (42), (43) and (44) into (41) we get

$$|\eta_M(-\sigma_2+it)| = O \left(\exp \left(\operatorname{vol}(M) \left(\frac{1}{2} + \sigma_2 \right) t + O(1) \right) \right) \quad \text{as } t \rightarrow +\infty. \quad (45)$$

Formula (6.1.45) from [1], which itself is an application of Stirling's formula, yields

$$\left(\frac{\Gamma(-\sigma_2+it)}{\Gamma(-\sigma_2-1/2+it)} \right)^{n_1} = O \left(t^{n_1/2} \right) \quad \text{as } t \rightarrow +\infty.$$

Therefore,

$$|K_M^{-1}(-\sigma_2+it)| = O(\exp \left(\frac{n_1}{2} \log t \right)) \quad \text{as } t \rightarrow +\infty. \quad (46)$$

Substituting the bound (46) together with (45) into (40) we get

$$\begin{aligned} |F(-\sigma_2+it)| &= O \left(\exp \left(\operatorname{vol}(M) \left(\frac{1}{2} + \sigma_2 \right) t + \frac{n_1}{2} \log t + O(\log(t)) \right) \right) \\ &\quad \cdot O \left(\exp \left(-\operatorname{vol}(M) \left(\frac{1}{2} + \sigma_2 + \delta \right) t \right) \right) \\ &= o(1) \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

One now can apply the Phragmen-Lindelöf theorem, which implies that $F(s) = O(1)$ in the sector $D := \{\pi/4 \leq \arg(s + \sigma_2) \leq \pi/2\}$ and the proof of a) is complete.

To prove (b), we repeat the proof given above for the function

$$G(s) = Z_M(s) \exp \left[\left(\frac{1}{2} + \sigma_2 \right) \operatorname{vol}(M) i s \right],$$

which is an entire function of finite order in the sector $D := \{\pi/4 \leq \arg(s + \sigma_2) \leq \pi/2\}$. Obviously, $|G(s)| = O(1)$ along the line $\arg(s + \sigma_2) = \pi/4$. To determine the behavior of the function $G(s)$ along the vertical line $\arg(s + \sigma_2) = \pi/2$, i.e. for $s = -\sigma_2 + it$, $t \geq 0$, we use the functional equation for the zeta function Z_M and get

$$\begin{aligned}
|G(-\sigma_2 + it)| &= \exp\left(-\left(\frac{1}{2} + \sigma_2\right) \text{vol}(M)t\right) |\eta_M(-\sigma_2 + it)| |\phi_M(1 + \sigma_2 - it)| |Z_M(1 + \sigma_2 - it)| \\
&= \exp\left(-\left(\frac{1}{2} + \sigma_2\right) \text{vol}(M)t\right) |\eta_M(-\sigma_2 + it)| |K_M(1 + \sigma_2 - it)| \cdot O(1) \quad \text{as } t \rightarrow +\infty,
\end{aligned}$$

since $1 + \sigma_2 \geq 2$ and the Dirichlet series for H_M and Z_M converge absolutely for $\text{Re } s \geq 2$ (see, e.g. page 160 of [34]). The estimate for $|\eta_M(-\sigma_2 + it)|$ is given by formula (45). Analogously as above, we see that

$$|K_M(1 + \sigma_2 - it)| = O(\exp\left(-\frac{n_1}{2} \log t\right)) \quad \text{as } t \rightarrow +\infty.$$

The proof is completed using the argument in the proof of part (a). ■

The following lemma is a Lindelöf type bound for the function $Z_M H_M$ which will be used to deduce a sharper bound for the function $\arg X_M(\sigma + iT)$, when σ is close to $1/2$.

Lemma 12 *For $\epsilon > 0$ and $t \geq 1$*

$$(Z_M H_M)\left(\frac{1}{2} + it\right) = O_\epsilon(\exp(\epsilon t)) \quad \text{as } t \rightarrow +\infty.$$

Proof. Let us write

$$\left|(Z_M H_M)\left(\frac{1}{2} + it\right)\right| = \left|Z_M\left(\frac{1}{2} + it\right)\right| \left|\phi_M\left(\frac{1}{2} + it\right)\right| \left|K_M^{-1}\left(\frac{1}{2} + it\right)\right|.$$

Since

$$\left|\phi_M\left(\frac{1}{2} + it\right)\right| = 1 \quad \text{and} \quad \left|K_M^{-1}\left(\frac{1}{2} + it\right)\right| = O(\exp(\frac{n_1}{2} \log t)) \quad \text{as } t \rightarrow +\infty,$$

it is enough to prove that

$$Z_M\left(\frac{1}{2} + it\right) = O_\epsilon(\exp(\epsilon t)) \quad \text{as } t \rightarrow +\infty,$$

We apply Proposition 6. In the notation of Proposition 6 we take

$$f(s) = Z_M(s) \quad \text{with} \quad \sigma_0 = \sigma_1 + \omega > \sigma_1 \quad \text{and} \quad P(t) = 2 \exp(t),$$

where σ_1 is defined in Lemma 9. Let us verify that all assumptions of Proposition 6 are fulfilled.

The function Z_M is meromorphic function of finite order, with poles at points on the real line; see page 498 of [29]. Hence $Z_M(s)$ is holomorphic function for $|\text{Im}(s)| \geq t_0 > 0$, for any $t_0 > 0$.

From the proof of Lemma 9 it is obvious that $|Z_M(s)| \geq c > 0$ and

$$Z'_M/Z_M(s) = O(1) \quad \text{as } t \rightarrow \infty,$$

for $s = \sigma + it$ and with $\sigma_0 - \omega \leq \sigma \leq \sigma_0 + \omega$. Furthermore, $|Z_M(s)| > 0$ for $\text{Re}(s) > \sigma_0 + \omega$.

From Lemma 11b), we have that

$$Z_M(\sigma + it) = O_\Gamma\left(\exp\left(\left(\frac{1}{2} + 3\sigma_0 - 2\right) \text{vol}(M)t\right)\right) = O_\Gamma\left(P(t)^D\right),$$

for a fixed $D = (3\sigma_0 - 3/2) \text{vol}(M)$, uniformly in $\sigma \geq 2 - 3\sigma_0$.

Theorem 5.3 from page 498 of [29] asserts that Z_M has no zeros in the half-plane $\operatorname{Re}(s) > 1/2$. Therefore, the Lindelöf condition on the vertical distribution of zeros of $Z_M(s)$ in the half-plane $\operatorname{Re}(s) > 1/2$, as required in Proposition 6, is trivially fulfilled.

Therefore, all the assumptions of Proposition 6 are satisfied, hence $Z_M(1/2 + it) = O_\epsilon(\exp(\epsilon t))$ as $t \rightarrow \infty$. ■

Lemma 13 *For an arbitrary $\epsilon > 0$, $t \geq 1$ and σ_2 defined in Lemma 11 we have*

$$(Z_M H_M)'(\sigma + it) = \begin{cases} O(\exp \epsilon t) & \text{for } \frac{1}{2} \leq \sigma \leq \sigma_0 \\ O(\exp(1/2 - \sigma + \epsilon)t) & \text{for } -\sigma_2 \leq \sigma < 1/2, \end{cases}$$

as $t \rightarrow \infty$.

Proof. The proof involves an application of the Phragmen-Lindelöf theorem to the open sector bounded by the lines

$$\operatorname{Re}(s) = -\sigma_2 \quad \text{and} \quad \operatorname{Re}(s) = \frac{1}{2}$$

and $\operatorname{Im}(s) = 1$. The bounds to be used come from Lemma 11, with $\delta = \epsilon$, and from Lemma 12. A direct application of the Phragmen-Lindelöf theorem yields the bound

$$(Z_M H_M)(\sigma + it) = O(\exp(1/2 - \sigma + \epsilon)t), \quad (47)$$

for $t \geq 1$ and $-\sigma_2 \leq \sigma \leq 1/2$. Similarly, for σ_0 defined as in Lemma 11, one can apply the Phragmen-Lindelöf theorem in the open sector bounded by the lines $\operatorname{Re}(s) = \sigma_0$, $\operatorname{Re}(s) = \frac{1}{2}$ and $\operatorname{Im}(s) = 1$, from which one gets

$$Z_M H_M(\sigma + it) = O(\exp \epsilon t), \quad (48)$$

for $1/2 \leq \sigma \leq \sigma_0$. The Cauchy integral formula can be applied, from which we have the equation

$$(Z_M H_M)'(s) = \frac{1}{2\pi i} \int_C \frac{(Z_M H_M)(z)}{(z - s)^2} dz$$

where C is a circle of a small, fixed radius $r < \epsilon$, centered at $s = \sigma + it$. Applying (48) to $(Z_M H_M)(z)$, when $1/2 \leq \operatorname{Re}(z) \leq \sigma_0$ and (47) when $\operatorname{Re}(z) < 1/2$, we deduce that

$$(Z_M H_M)'(\sigma + it) = O(\exp((r + \epsilon)t/r)) = O(\exp(2\epsilon t))$$

for $1/2 \leq \sigma \leq \sigma_0$ and $t \geq 1$. This proves the first part of the Lemma when replacing ε by $\varepsilon/2$.

In the case when $\sigma < 1/2$, we can use the functional equation for $(Z_M H_M)'$ to arrive at the expression

$$\begin{aligned} |(Z_M H_M)'(-\sigma_2 + it)| &= |\eta_M(-\sigma_2 + it)| |K_M^{-1}(-\sigma_2 + it)| |Z_M(1 + \sigma_2 - it)| \cdot \\ &\quad \cdot \left| \frac{\eta'_M}{\eta_M}(-\sigma_2 + it) - \frac{K'_M}{K_M}(-\sigma_2 + it) - \frac{Z'_M}{Z_M}(1 + \sigma_2 - it) \right|. \end{aligned}$$

Since $\sigma_2 \geq 1$, we have $Z'_M/Z_M(1 + \sigma_2 - it) = O(1)$, as $t \rightarrow +\infty$. Elementary computations involving the definition of the function η'_M/η_M and the Stirling formula imply that

$$\frac{\eta'_M}{\eta_M}(-\sigma_2 + it) - \frac{K'_M}{K_M}(-\sigma_2 + it) - \frac{Z'_M}{Z_M}(1 + \sigma_2 - it) = O(t) \quad \text{as } t \rightarrow \infty;$$

in brief, one sees the asymptotic bound by observing that the leading term in the above expression is $\text{vol}(M)(1/2 + \sigma_2 - it) \tan(\pi(1/2 + \sigma_2 - it))$. From the bounds (45) and (46) obtained in the proof of Lemma 11, we arrive at the bound

$$|(Z_M H_M)'(-\sigma_2 + it)| = O\left(\exp\left(\left(\frac{1}{2} + \sigma_2 + \epsilon\right) \text{vol}(M)t\right)\right) \quad \text{as } t \rightarrow \infty.$$

The bound claimed in the statement of the Lemma follows by applying the Phragmen-Lindelöf theorem to the function $(Z_M H_M)'$ in the open sector bounded by the lines $\text{Im}(s) = 1$, $\text{Re}(s) = -\sigma_2$ and $\text{Re}(s) = 1/2$, keeping in mind that $-\sigma_2 \leq \sigma < 1/2$. ■

5 Vertical and horizontal distribution of zeros

In this section we will prove parts b) and c) of the Main Theorem.

We fix a large positive number T and choose number T' such that $|T' - T| = O(1)$ independently of T where no zero of $Z_M H_M$ has imaginary part equal to T' . Let $t_0 > 0$ be a number such that $(Z_M H_M)'/(Z_M H_M)(\sigma + it) \neq 0$ for all $\sigma < 1/2$ and $|t| < t_0$; the existence of such t_0 is established by Proposition 8. Let $\sigma_0 \geq 1$ be a constant chosen so that $\sigma_0 \geq \max\{\sigma'_0, \sigma_1\}$, where σ'_0 is defined in Lemma 1 and σ_1 is defined in Lemma 9. Let $0 < a < 1/2$ be arbitrary.

The function $X_M(s)$, which was defined in (38), is holomorphic in the rectangle $R(a, T')$ with vertices $a + it_0$, $\sigma_0 + it_0$, $\sigma_0 + iT'$ and $a + iT'$. As in [44], we will use Littlewood's theorem, as stated in section 2.8, from which we get the formula

$$\begin{aligned} 2\pi \sum_{\substack{\rho' = \beta' + i\gamma \\ t_0 < \gamma < T', \beta' > a}} (\beta' - a) &= \int_{t_0}^{T'} \log |X_M(a + it)| dt - \int_{t_0}^{T'} \log |X_M(\sigma_0 + it)| dt \\ &\quad - \int_a^{\sigma_0} \arg X_M(\sigma + it_0) d\sigma + \int_a^{\sigma_0} \arg X_M(\sigma + iT') d\sigma = I_1 - I_2 - I_3 + I_4. \end{aligned} \quad (49)$$

The variable ρ' denotes a zero of $(Z_M H_M)'$, and the integrals I_1 , I_2 , I_3 and I_4 are defined to be the four integrals in (49), in obvious notation. By Proposition 8, the condition that $\text{Im}(\rho') > t_0$ implies that $\text{Re}(\rho') \geq 1/2$, hence the sum on the left hand side of (49) is actually taken over all zeros of $(Z_M H_M)'$ with imaginary part in the interval (t_0, T') .

We investigate integrals I_1 , I_2 , I_3 and I_4 separately.

Obviously, $I_3 = O(1)$ as $T \rightarrow \infty$ since, in fact, I_3 is independent of T . As for I_2 , we will follow the argument from page 1144 of [44]. Let us write

$$I_2 = \int_{t_0}^{T'} \log X_M(\sigma_0 + it) dt + \int_{t_0}^{T'} \arg X_M(\sigma_0 + it) dt.$$

The function $X_M(s)$ is holomorphic and non-vanishing in a half-plane which contains the line of integration, so

$$\int_{t_0}^{T'} \arg X_M(\sigma_0 + it) dt = O(1) \quad \text{as } T \rightarrow \infty.$$

By Cauchy's theorem,

$$\int_{t_0}^{T'} \log X_M(\sigma_0 + it) dt = \int_{\sigma_0}^{\infty} \log X_M(\sigma_0 + iT') d\sigma - \int_{\sigma_0}^{\infty} \log X_M(\sigma + it_0) d\sigma,$$

which is bounded in T' since the function $\log X_M$ is holomorphic and bounded in the infinite strip $\{s \in \mathbb{C} : t_0 \leq \text{Im}(s) \leq T', \text{Re}(s) \geq \sigma_0\}$.

It remains to evaluate I_1 and I_4 .

5.1 Evaluation of I_1

We shall break apart further I_1 by using the functional equation (19) for $(Z_M H_M)'$, the definition (38) of X_M , and representation of $\tilde{Z}_M(s) = 1 + Z_{M,1}(s)$ which was used in the proof of Proposition 7b). By doing so, we arrive at the expression

$$\begin{aligned} I_1 = & - \int_{t_0}^{T'} \log |a_M A_M^{-(a+it)}| dt + \int_{t_0}^{T'} \log |f_M(a+it) \eta_M(a+it) K_M^{-1}(a+it)| dt \\ & + \int_{t_0}^{T'} \log |Z_M(1-(a+it))| dt + \int_{t_0}^{T'} \log |1 + Z_{M,1}(1-(a+it))| dt = I_{11} + I_{12} + I_{13} + I_{14}, \end{aligned}$$

with the obvious notation for the integrals I_{11} , I_{12} , I_{13} and I_{14} . Clearly, we have that

$$I_{11} = -T(\log |a_M| - a \log A_M) + O(1) \quad \text{as } T \rightarrow \infty. \quad (50)$$

From the computations on the bottom of page 1146 of [44], we have that

$$\int_{t_0}^{T'} \log |f_M(a+it)| dt = T \log T + T(\log \text{vol}(M) - 1) + O(\log T) \quad \text{as } T \rightarrow \infty. \quad (51)$$

Therefore,

$$\begin{aligned} I_{12} = & T \log T + T(\log \text{vol}(M) - 1) + O(\log T) \\ & + \int_{t_0}^{T'} \log |\eta_M(a+it)| dt + \int_{t_0}^{T'} \log |K_M^{-1}(a+it)| dt \\ = & T \log T + T(\log \text{vol}(M) - 1) + I_{121} + I_{122} + O(\log T) \quad \text{as } T \rightarrow \infty, \end{aligned} \quad (52)$$

with obvious notation for I_{121} and I_{122} . Stirling's formula implies that

$$|K_M^{-1}(a+it)| = \pi^{-\frac{n_1}{2}} \exp(-c_1 a - \text{Re}(c_2)) \exp \left[n_1 \left(\frac{1}{2} \log |a - 1/2 + it| + O\left(\frac{1}{t}\right) \right) \right] \left(1 + O\left(\frac{1}{t^2}\right) \right),$$

as $t \rightarrow \infty$, where c_1 and c_2 are constants defined in (6). Therefore,

$$\begin{aligned}
I_{122} &= \int_{t_0}^{T'} \log |K_M^{-1}(a+it)| dt = \left(-c_1 a - \operatorname{Re}(c_2) - \frac{n_1}{2} \log \pi \right) T \\
&\quad + \frac{n_1}{2} \int_{t_0}^{T'} \log |a - 1/2 + it| + O(\log T) \\
&= \frac{n_1}{2} T \log T - T \left(c_1 a + \operatorname{Re} c_2 + \frac{n_1}{2} \log \pi + \frac{n_1}{2} \right) + O(\log T) \quad \text{as } T \rightarrow \infty.
\end{aligned} \tag{53}$$

Finally,

$$\begin{aligned}
I_{121} &= \int_{t_0}^{T'} \operatorname{Re} \left(\int_{1/2}^{a+it} \frac{\eta'_M}{\eta_M}(u) du \right) dt = \int_{t_0}^{T'} \operatorname{Re} \left(\int_{1/2}^{a+it} \operatorname{vol}(M)(u - 1/2) \tan \pi(u - 1/2) du \right) dt \\
&\quad + 2n_1 \log 2 (a - 1/2) (T' - t_0) + n_1 \int_{t_0}^{T'} \operatorname{Re} \left(\int_{1/2}^{a+it} \left(\frac{\Gamma'}{\Gamma}(\frac{1}{2} + u) + \frac{\Gamma'}{\Gamma}(\frac{3}{2} - u) \right) du \right) dt \\
&\quad + \operatorname{Re} \left(- \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{\pi}{M_R \sin \theta} \int_{1/2}^{a+it} \frac{\cos(2\theta - \pi)(s - 1/2)}{\cos \pi(s - 1/2)} ds \right) \\
&= \left(\frac{1}{2} - a \right) \frac{\operatorname{vol}(M)}{2} T^2 + 2n_1 \log 2 (a - 1/2) T + O(\log T) \quad \text{as } T \rightarrow \infty.
\end{aligned} \tag{54}$$

As in the proof of Lemma 11, one can use Stirling's formula to obtain the bound

$$\int_{t_0}^{T'} \operatorname{Re} \left(\int_{1/2}^{a+it} \left(\frac{\Gamma'}{\Gamma}(\frac{1}{2} + u) + \frac{\Gamma'}{\Gamma}(\frac{3}{2} - u) \right) du \right) dt = O(\log T) \quad \text{as } T \rightarrow \infty.$$

Elementary computations yield the same bound for the elliptic contribution. By substituting (54) and (53) into (52), we arrive at the bound

$$\begin{aligned}
I_{12} &= \left(\frac{1}{2} - a \right) \frac{\operatorname{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + 1 \right) T \log T + O(\log T) \\
&\quad + T \left[2n_1 \log 2 (a - 1/2) - c_1 a + \log \operatorname{vol}(M) - 1 - \operatorname{Re}(c_2) - \frac{n_1}{2} \log \pi - \frac{n_1}{2} \right] \quad \text{as } T \rightarrow \infty.
\end{aligned} \tag{55}$$

With all this, we arrive at the desired bound for I_{12} .

The integral I_{13} is estimated by applying the Cauchy's theorem to the function $\log Z_M(s)$ within in the rectangle with vertices $1 - a - iT'$, $2 - iT'$, $2 - it_0$ and $1 - a - it_0$. As in [44], it easily is shown that

$$I_{13} = - \int_{1-a}^2 \arg Z_M(\sigma - iT') d\sigma + O(1) = O \left(\max_{1-a \leq \sigma \leq 2} |\log Z_M(\sigma - iT')| \right).$$

From

$$\log Z_M(\sigma - iT') = \log Z_M(2 - iT') - \int_{\sigma - iT'}^{2 - iT'} \frac{Z'_M}{Z_M}(\xi) d\xi,$$

and the bound in (25), which we write as

$$\frac{Z'_M}{Z_M}(\alpha \pm iT') = O((T \log T)^{2-2\alpha}) \quad \text{for } 1 - a \leq \alpha < 1/2,$$

we obtain the expression

$$I_{13} = \int_{t_0}^{T'} \log |Z_M(1 - a - it)| dt = O((T \log T)^{2-2(1-a)}) = O((T \log T)^{2a}) \quad \text{as } T \rightarrow \infty. \quad (56)$$

Directly from Lemma 10, we have the estimate

$$I_{14} = \int_{t_0}^{T'} \log |1 + Z_{M,1}(1 - (a + it))| dt = O((T \log T)^{2a}) \quad \text{as } T \rightarrow \infty. \quad (57)$$

Combining (50), (55), (56) and (57) yields

$$I_1 = \left(\frac{1}{2} - a\right) \frac{\text{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + 1\right) T \log T + O((T \log T)^{2a}) + TC_{M,a} \quad \text{as } T \rightarrow \infty, \quad (58)$$

where

$$C_{M,a} = (a - 1/2) \cdot 2n_1 \log 2 + a(\log A_M - c_1) - \log |a_M| + \log \text{vol}(M) - 1 - \text{Re}(c_2) - \frac{n_1}{2} \log \pi - \frac{n_1}{2}.$$

Finally, we have arrived at our estimate for I_1 .

5.2 Evaluation of I_4

The evaluation of I_4 closely follows the lines of the proof that the analogous integral in the compact case considered by Garunkštis in [25]. The new input being our Lemma 13.

It is sufficient to prove that

$$\arg X(\sigma + iT') = o(T) \quad \text{for } a \leq \sigma \leq \sigma_0 \text{ and as } T \rightarrow \infty. \quad (59)$$

We first will show that (59) holds when $1/2 - \delta \leq \sigma \leq \sigma_0$ for some small δ . The argument is similar to the proof of the formula (3.4) in [25]. However, in order to keep the exposition self-contained, we will repeat main steps of the proof.

Let $\delta > 0$ be a small constant to be chosen later and let N_δ denote the number of zeros of $\text{Re}(X_M(\sigma + iT'))$ with $1/2 - \delta \leq \sigma \leq \sigma_0$. Then $|\arg X_M(\sigma + iT')| \leq \pi(N_\delta + 1)$. In order to estimate N_δ , consider the function

$$g(z) := 1/2(X_M(z + iT') + X_M(z - iT')).$$

Let $n(r, \sigma_0)$ denotes the number of zeros of $g(z)$ inside the circle $|z - \sigma| \leq r$. Observe that

$$g(\sigma) = \operatorname{Re}(X_M(\sigma + iT')) \quad \text{and} \quad |\arg X_M(\sigma + iT')| \leq \pi(n(\sigma_0 - 1/2 + \delta, \sigma_0) + 1).$$

By applying Jensen's theorem to g , we obtain the equation

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(\sigma_0 + Re^{i\theta})| d\theta - \log |g(\sigma_0)|. \quad (60)$$

For large enough T and $R = \sigma_0 - 1/2 + 2\delta$, Lemma 13 implies the bound

$$\log |g(\sigma_0 + re^{i\theta})| < \begin{cases} \epsilon T, & \text{for } \operatorname{Re}(\sigma_0 + Re^{i\theta}) \geq \frac{1}{2} \\ (2\delta + \epsilon)T, & \text{for } \frac{1}{2} - 2\delta \leq \operatorname{Re}(\sigma_0 + Re^{i\theta}) \leq \frac{1}{2}. \end{cases}$$

The circle $|z - \sigma_0| = R$ has a very small part to the left of the line $\operatorname{Re}(s) = 1/2$ namely a circular arc of length $O(\delta^{1/2})$. Therefore, the right hand side of (60) is $O(\epsilon T) + O(\delta^{1/2}(2\delta + \epsilon)T)$. Hence,

$$n(R - \delta) \leq \frac{R}{\delta} \int_0^R \frac{n(r)}{r} dr = O\left(\left(\frac{\epsilon}{\delta} + \frac{\delta + \epsilon}{\sqrt{\delta}}\right)T\right).$$

For $a \leq \sigma < 1/2 - \delta$ we use Proposition 8 to deduce that $\operatorname{Re}(X(\sigma + iT')) \neq 0$. In particular, we have that $\arg X(\sigma + iT') = O(1)$ when $a \leq \sigma < 1/2 - \delta$ since \arg is bounded by π times the number of zeros.

Let us now take $\delta = \epsilon^{2/3}$. We then get the bound

$$|\arg X_M(\sigma + iT')| \leq \pi(O(\epsilon^{1/3}T) + 1) + O(1) = o(T) \quad \text{as } T \rightarrow \infty,$$

for all $a \leq \sigma \leq \sigma_0$. Since $\epsilon > 0$ is arbitrary, we conclude that $I_4 = o(T)$ as $T \rightarrow \infty$.

5.3 Proof of the Main Theorem

Since $0 < a < 1/2$, we have that $(T \log T)^{2a} = o(T)$. We have shown that I_2 and I_3 are $O(1)$ as $T \rightarrow \infty$ and that $I_4 = o(T)$ as $T \rightarrow \infty$. Hence, by substituting equation (58) into (49) we get

$$2\pi \sum_{\substack{\rho' = \beta' + i\gamma \\ t_0 < \gamma < T'}} (\beta' - a) = \left(\frac{1}{2} - a\right) \frac{\operatorname{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + 1\right) T \log T + TC_{M,a} + o(T) \quad \text{as } T \rightarrow \infty, \quad (61)$$

where

$$C_{M,a} = (a - 1/2) \cdot 2n_1 \log 2 + a(\log A_M - c_1) - \log |a_M| + \log \operatorname{vol}(M) - 1 - \operatorname{Re}(c_2) - \frac{n_1}{2} \log \pi - \frac{n_1}{2}.$$

Substituting $a/2$ instead of a into 61, subtracting the obtained formulas, and then dividing by $a/2$ yields the statement b) of the Main Theorem.

As for part (c) of the Main Theorem, we begin with the formula

$$\sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma\leq T}} (\beta' - 1/2) = \sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma<T'}} (\beta' - a) + (a - 1/2) \sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma<T'}} 1. \quad (62)$$

The first sum on the right hand side of (62) is estimated by (61). The second sum on the right hand side of (62) is estimated by part b) of the Main Theorem, keeping mind that the difference between the second sum in (62) and the sum in part b) is the finite number of zeros in the half-plane $\text{Re}(s) < 1/2$.

With all this, the proof of the Main Theorem is complete.

In the case when the surface is co-compact the statement of the Main Theorem is easily deduced, since, in that case $n_1 = c_1 = c_2 = 0$, $H_M = 1$, $A_M = \exp(\ell_{M,0})$ and

$$\frac{\eta'_M(s)}{\eta_M(s)} = \text{vol}(M)(s - 1/2) \tan(\pi(s - 1/2)) - \pi \sum_{\substack{\{R\} \\ 0 < \theta(R) < \pi}} \frac{1}{M_R \sin \theta} \frac{\cos(2\theta - \pi)(s - 1/2)}{\cos \pi(s - 1/2)}.$$

6 Corollaries of the Main Theorem

In this section we deduce three corollaries of our Main Theorem. The results we prove are analogous to Theorem 2 and Theorem 3 in [42], with, in their notation, $k = 1$. Similar results may be deduced for the horizontal distribution of zeros of the k th derivative, based on the results of Section 8, with suitably replaced constants.

Corollary 14 *For $\delta > 1/2$, let $N_{\text{ver}}(\delta, T; (Z_M H_M)')$ denote the number of zeros ρ' of $(Z_M H_M)'$ such that $\text{Re}(\rho') > \delta$ and $0 < \text{Im}(\rho') < T$. Then, for an arbitrary $\epsilon > 0$*

$$N_{\text{ver}}\left(\frac{1}{2} + \epsilon, T; (Z_M H_M)'\right) < \frac{1}{\epsilon} N_{\text{hor}}(T; (Z_M H_M)').$$

Proof. Trivially, we have the bounds

$$N_{\text{ver}}\left(\frac{1}{2} + \epsilon, T; (Z_M H_M)'\right) < \frac{1}{1/2 + \epsilon} \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma > 1/2 + \epsilon, 0 < t < T}} \sigma \quad (63)$$

$$= \frac{1}{1/2 + \epsilon} \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma > 1/2 + \epsilon, 0 < t < T}} \left(\sigma - \frac{1}{2}\right) + \frac{1/2}{1/2 + \epsilon} N_{\text{ver}}\left(\frac{1}{2} + \epsilon, T; (Z_M H_M)'\right). \quad (64)$$

Therefore,

$$\frac{2\epsilon}{1 + 2\epsilon} N_{\text{ver}}\left(\frac{1}{2} + \epsilon, T; (Z_M H_M)'\right) < \frac{2}{1 + 2\epsilon} N_{\text{hor}}(T; (Z_M H_M)'),$$

from which the result immediately follows. ■

Observe that the lead term in the asymptotic expansion in part (b) of the Main Theorem is $O(T^2)$, whereas the lead term in the asymptotic expansion in part (c) of the Main Theorem is $O(T \log(T))$. Consequently, Corollary 14 shows that zeros of $(Z_M H_M)'$ are concentrated very close the critical line $\text{Re}(s) = 1/2$. The following corollary further quantifies this observation.

Corollary 15 For any $\delta > 1/2$, let $N_{\text{ver}}^-(\delta, T; (Z_M H_M)')$ denote the number of non-trivial zeros $\rho = \sigma + it$ of $(Z_M H_M)'$ with $\sigma < \delta$ and $0 < t < T$. Then, for any constant $\epsilon > 0$,

$$\lim_{T \rightarrow \infty} \frac{N_{\text{ver}}^-(1/2 + \epsilon, T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')} = 1.$$

Proof. Obviously,

$$1 \geq \frac{N_{\text{ver}}^-(1/2 + \epsilon, T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')} = 1 - \frac{N_{\text{ver}}(1/2 + \epsilon, T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')} + \frac{O(1)}{N_{\text{vert}}(T; (Z_M H_M)')},$$

where the last term represents the contribution from at most finitely many non-trivial zeros of $(Z_M H_M)'$ in the half-plane $\text{Re}(s) < 1/2$. Corollary 14 implies that

$$1 \geq \frac{N_{\text{ver}}^-(1/2 + \epsilon, T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')} > 1 - \frac{1}{\epsilon} \frac{N_{\text{hor}}(T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')}. \quad (65)$$

From the Main Theorem b) and c) we deduce that

$$\frac{N_{\text{hor}}(T; (Z_M H_M)')}{N_{\text{vert}}(T; (Z_M H_M)')} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Therefore, by passing to the limit as $T \rightarrow \infty$ in (65), the claimed result follows. ■

A result similar to Corollary 15, for the zeros of the derivative of the Selberg zeta function associated to a compact Riemann surface is obtained in [47].

The following corollary gives estimates of short sums of distances $(\sigma - 1/2)$.

Corollary 16 Let $0 < U < T$. Then,

$$2\pi \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma > 1/2, \ T < t \leq T+U}} \left(\sigma - \frac{1}{2} \right) = \left(\frac{n_1}{2} + 1 \right) U \log(T + U) \quad (66)$$

$$+ \left(\log \frac{\mathbf{g}_1 \text{vol}(M) A_M^{1/2}}{|\pi^{n_1/2} d(1) a_M|} \right) U + o(T) + O(U^2/T) \quad \text{as } T \rightarrow \infty. \quad (67)$$

Proof. The left hand side of the (66) is equal to $2\pi(N_{\text{hor}}(T + U; (Z_M H_M)') - N_{\text{hor}}(T; (Z_M H_M)'))$, hence part (c) of the Main Theorem yields

$$2\pi \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma > 1/2, \ T < t \leq T+U}} \left(\sigma - \frac{1}{2} \right) = \left(\frac{n_1}{2} + 1 \right) \left(T \log \left(1 + \frac{U}{T} \right) - U \right) \quad (68)$$

$$+ \left(\log \frac{\mathbf{g}_1 \text{vol}(M) A_M^{1/2}}{\pi^{n_1/2} |d(1) a_M|} \right) U + o(T) \quad \text{as } T \rightarrow \infty. \quad (69)$$

The elementary observation that $T \log(1 + \frac{U}{T}) - U = O(U^2/T)$ completes the proof. ■

7 Examples

The Main Theorem naturally leads to the following question: Are there examples of groups Γ where $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ as well as groups where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$? The purpose of this section is to present examples of groups in each category. In fact, there are examples of both arithmetic and non-arithmetic groups in each category.

7.1 Congruence subgroups

Let $\Gamma = \overline{\Gamma_0(N)}$ be the congruence subgroups defined by the arithmetic condition

$$\overline{\Gamma_0(N)} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\} / \pm I,$$

where I denotes the identity matrix and N is a squarefree, positive integer. If $N = p_1 \cdots p_r$, for distinct primes p_1, \dots, p_r ; then, it is proved in [29], pages 532-538, as well as in [32], that the corresponding surface has $n_1 = 2^r$ cusps and the scattering determinant is given by the formula

$$\varphi_N(s) = \left[\sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \right]^{n_1} \left[\frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)} \right]^{n_1} \prod_{p|N} \left(\frac{1-p^{2-2s}}{1-p^{2s}} \right)^{n_1/2}$$

Now, it is easy to show that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$.

All elements of $\overline{\Gamma_0(N)}$ have integer entries, so any hyperbolic element has trace whose absolute value is at least equal to 3. Therefore, $e^{\ell_{M,0}} \geq u$ where u is a solution to $u^{1/2} + u^{-1/2} = 3$. Solving, we get that $u = ((3 + \sqrt{5})/2)^2 > 4$. Therefore, for any such group $\overline{\Gamma_0(N)}$, one has that $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Further examples of arithmetic Fuchsian groups where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$ are the groups $\overline{\Gamma(N)}$, where $\Gamma(N)$ denotes the principal congruence subgroup. The scattering determinant can be computed using the analysis presented in [29] and [32]. As above, one shows that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$ because the Dirichlet series portion of the scattering determinant is shown to give by ratios of classical Dirichlet series. Furthermore, the matrices in $\Gamma(N)$ also have integral entries, so $e^{\ell_{M,0}} \geq ((3 + \sqrt{5})/2)^2 > 4$.

7.2 Moonshine subgroups

We now present an example of a non-compact, arithmetic Riemann surface where $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Following [21], we will use the term "moonshine group" for any subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ which satisfies the following two conditions. First, there exists an integer $N \geq 1$ such that Γ contains $\overline{\Gamma_0(N)}$. Second, Γ contains the element

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{if and only if} \quad k \in \mathbb{Z}.$$

Such groups also appear in [54], defined as $\overline{\Gamma_0^+(N)} := \overline{\Gamma_0(N)} \cup \overline{\Gamma_0(N)}\tau$, where

$$\tau = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We employ the term "moonshine groups" following the discussion in [11], [13], [14], [15] and [22]. Specifically, we refer to Proposition 7.1.2, page 408 of [22] which cites results from [11] regarding a

classification of genus zero subgroups of $\mathrm{SL}(2, \mathbb{R})$ which are manifest in the “monstrous moonshine” conjectures that were proved by Borchers.

In this section we will examine two of the genus zero “moonshine groups” which were determined in [11], showing that for one group one has that $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$ and for another one has that $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$. In fact, the two groups have the additional feature of having the same topological signature.

Following [30] as well as page 363 of [14], let f be a square-free, non-negative integer, and consider the group

$$\Gamma_0(f)^+ := \left\{ e^{-1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) : a, b, c, d, e \in \mathbb{Z}, e \mid f, e \mid a, e \mid d, f \mid c, ad - bc = e \right\}$$

In [30], it is proved that if a subgroup $G \subseteq \mathrm{SL}(2, \mathbb{R})$ is commensurable with $\mathrm{SL}(2, \mathbb{Z})$, then there exists a square-free, non-negative integer f such that G is a subgroup of $\Gamma_0(f)^+$. Lemma 2.20 on page 368 of [14] proves that the parabolic elements of $\Gamma_0(f)^+$ have integral entries. Therefore, $\Gamma_0(f)^+$ is a moonshine group. Let $\overline{\Gamma_0(f)^+} = \Gamma_0(f)^+ / \pm I$. The Riemann surface $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$ has finite volume since the fundamental domain has smaller area than the fundamental domain of $\Gamma_0(f)$. Since all parabolic elements of $\Gamma_0(f)^+$ have integral entries and cusps of the corresponding surface are uniquely determined by the parabolic elements, we conclude that the surface $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$ has at most $n_1 = 2^r$ inequivalent cusps, where r is the number of prime factors of f . The number of inequivalent cusps of the surface $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$ may be strictly less than 2^r , as we will see in the following example. Heuristically, this is expected, since the surface $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$ has a “smaller” fundamental polygon.

Consider the case when $f = 5$. As proved in [14], the surface $\overline{\Gamma_0(5)^+} \backslash \mathbb{H}$ has one cusp at ∞ . The scattering matrix in this case has a single entry which is given by

$$\Phi_5(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \varphi_{5,\infty\infty}(s),$$

for $\mathrm{Re}(s) \gg 0$ and where

$$\varphi_{5,\infty\infty}(s) = \sum_{c \in C_5} c^{-2s} S(c),$$

with

$$C_5 = \left\{ c > 0 : c = 5n \text{ or } c = \sqrt{5} \cdot m, \text{ for some positive integers } n, m, \text{ such that } 5 \nmid m \right\}$$

and $S(c)$ denotes the number of distinct numbers $d \bmod(c)$ such that d is the right lower entry of the matrix from $\Gamma_0(5)^+$ whose left lower entry is $c \in C_5$. We refer to [34], Sections 2.5 and 3.4 for details. In [38] it is proved that

$$\varphi_{5,\infty\infty}(s) = \left(\frac{5^s + 5}{5^s(5^s + 1)} \right) \cdot \frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)}.$$

With all this, one immediately can show that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$.

It is easily to confirm that

$$\gamma = \begin{pmatrix} 0 & -1/\sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{pmatrix} \in \Gamma_0(5)^+,$$

which is seen by taking $e = 5$, $a = 0$, $b = -1$, and $c = d = 5$. The trace of γ is $\sqrt{5} > 2$, hence γ is hyperbolic. Therefore, $e^{\ell_{M,0}} \leq u$ where u is a positive solution of $u^{1/2} + u^{-1/2} = \sqrt{5}$. Solving, we have that $u = ((1 + \sqrt{5})/2)^2 < 4$.

With all this, we have an example of an arithmetic Riemann surface where $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Since the surface $\overline{\Gamma_0(5)^+} \backslash \mathbb{H}$ has a signature $(0;2,2,2;1)$ meaning that its genus is zero, it has three inequivalent elliptic points of order two and one cusp. With this, a natural question to consider is if the inequality $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ holds for all surfaces of signature $(0;2,2,2;1)$. The answer to this question is no, as we will show in the following example.

The surface $\overline{\Gamma_0(6)^+} \backslash \mathbb{H}$ has the signature $(0;2,2,2;1)$, as shown in Table C of [14]. The scattering matrix in this case has a single entry which is given by

$$\Phi_6(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \varphi_{6,\infty}(s),$$

for $\text{Re}(s) \gg 0$, where

$$\varphi_{6,\infty}(s) = \sum_{c \in C_6} c^{-2s} S(c),$$

with

$$\begin{aligned} C_6 = \{ & c > 0 : c = 6n \text{ or } c = 3\sqrt{2} \cdot n, \text{ or } c = 2\sqrt{3} \cdot n, \\ & \text{or } c = \sqrt{6} \cdot m \text{ for integers } n, m > 0, \text{ such that } 6 \nmid m \} \end{aligned} \quad (70)$$

and $S(c)$ denotes the number of distinct numbers $d \bmod(c)$ such that d is the right lower entry of the matrix from $\Gamma_0(6)^+$ whose left lower entry is $c \in C_6$. Obviously, $\mathfrak{g}_1 = \sqrt{6}$ and $\mathfrak{g}_2 = 2\sqrt{3}$ are two smallest elements of C_6 , hence $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 2$. On the other hand, traces of matrices from $\Gamma_0(6)^+$ belong to the set $T = \{(a+d) \cdot h, \text{ where } a, d \in \mathbb{Z} \text{ and } h \in \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}\}$. Therefore, $\min\{|\text{Tr} A| : A \in \mathcal{H}(\Gamma_0(6)^+)\} = \sqrt{6}$, hence $e^{\ell_{M,0}} \geq u$ where $u > 1$ is a solution of the equation $u^{1/2} + u^{-1/2} = \sqrt{6}$. Since $u = ((\sqrt{6} + \sqrt{2})/2)^2 > 2$, we see that the surface $\overline{\Gamma_0(6)^+} \backslash \mathbb{H}$ is an example of the surface where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

More generally, if $f \geq 5$ is a prime number, then the surface $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$ has one cusp at infinity and the corresponding set C has the form $C = \{c > 0 : c = f \cdot n \text{ or } c = \sqrt{f} \cdot n, \text{ for some integer } n > 0\}$, hence $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$ for all surfaces $\overline{\Gamma_0(f)^+} \backslash \mathbb{H}$, where $f \geq 5$ is prime. Furthermore, if $f \geq 7$ is a prime, then, the group $\Gamma_0(f)^+$ does not contain a hyperbolic element whose corresponding geodesic has hyperbolic length less than $\log 4$. So, groups $\Gamma_0(f)^+$ with $f \geq 7$ prime yield to yet another example of the Riemann surface where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

We refer the reader to the article [38] for detail concerning the computations of the scattering matrices for the "moonshine" groups discussed in this section, as well as further analytic and numerical investigations of the distribution of eigenvalues of the Laplacian.

7.3 On existence of surfaces where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$

We now argue the existence of an abundance of surfaces for which $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Let M_τ denote a degenerating family of Riemann surfaces, parameterized by the holomorphic parameter τ , which approach the Deligne-Mumford boundary of moduli space when τ approaches zero. One can select distinguished points of M_τ which are either removed or whose local coordinates z are replaced by fractional powers $z^{1/n}$. By doing so, one obtains a degenerating sequence of

hyperbolic Riemann surfaces of any signature; we refer the reader to [31] and references therein for further details regarding the construction of the sequence of degenerating hyperbolic Riemann surfaces.

By construction, the length of the smallest geodesic on M_ℓ approaches zero, so then $\exp(\ell_{M_\tau,0})$ approaches one as τ approaches zero. In [23], the authors prove that through degeneration, parabolic Eisenstein series on M_τ converge to parabolic Eisenstein series on the limit surface; see part (ii) of the Main Theorem on page 703 of [23]. To be precise, one needs that the holomorphic parameter s of the parabolic Eisenstein series lies in the half-plane $\text{Re}(s) > 1$ and the spacial parameter z to lie in a bounded region of M_τ . However, in these ranges, one can compute the scattering matrix by computing the zeroth Fourier coefficient of the parabolic Eisenstein series, and, subsequently, compute the ratio $\mathfrak{g}_2/\mathfrak{g}_1$ on M_τ . Since the parabolic Eisenstein series converge through degeneration to the parabolic Eisenstein series the limit surface, the associated scattering matrix converges to a submatrix Φ of the full scattering matrix on the limit surface. Clearly, the determinant of Φ can be decomposed into a product of Gamma functions and a Dirichlet series, where the Dirichlet series is such that $\mathfrak{g}_2/\mathfrak{g}_1 > 1$.

Therefore, we conclude that for all τ sufficiently close to zero, we have that $e^{\ell_{M_\tau,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$. In fact, all surfaces near the Deligne-Mumford boundary of any given moduli space satisfy the inequality $e^{\ell_{M_\tau,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

Additionally, let us assume that one is considering a moduli space which contains a congruence subgroup so then there exists a surface where $e^{\ell_{M_\tau,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$. Then by combining the above argument with the computations from section 7.1, there exists surfaces for which $e^{\ell_{M_\tau,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. However, we have not been successful in our attempts to explicitly construct such a surface. In a sense, our Main Theorem shows that surfaces for which $e^{\ell_{M_\tau,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$ have a larger number of zeros of $(Z_M H_M)'$ than nearby surfaces for which the inequality holds.

8 Higher derivatives

In this section, we will outline the proof of the Main Theorem for higher order derivatives of $Z_M H_M$. The results are analogous to theorems proved for the zeros of the higher order derivatives of the Riemann zeta function; see [7] and [42].

8.1 Preliminary lemmas on higher derivatives

In order to deduce the vertical and horizontal distribution of zeros of the higher order derivatives of $(Z_M H_M)$ we prove some preliminary lemmas, analogous to lemmas in Section 4.

Lemma 17 *Let $f_M(s)$ be defined by (20) and $\tilde{Z}_M(s)$ defined by (21). Let us define, inductively, the functions $\tilde{Z}_{M,j}(s)$ as $\tilde{Z}_{M,0}(s) := Z_M(s)$, $\tilde{Z}_{M,1}(s) := \tilde{Z}_M(s)$ and, for $j \geq 2$,*

$$\tilde{Z}_{M,j}(1-s) = \frac{1}{f_M(s)} \left((j-1) \frac{f'_M(s)}{f_M(s)} + \frac{\eta'_M(s)}{\eta_M(s)} - \frac{K'_M(s)}{K_M(s)} - \sum_{i=0}^{j-1} \frac{\tilde{Z}'_{M,i}(s)}{\tilde{Z}_{M,i}(s)} (1-s) \right). \quad (71)$$

Then for every positive integer k the k th derivative of the function $Z_M H_M$ can be represented as

$$(Z_M H_M)^{(k)}(s) = (f_M(s))^k \eta_M(s) K_M^{-1}(s) Z_M(1-s) \prod_{i=1}^k \tilde{Z}_{M,i}(1-s). \quad (72)$$

Proof. The statement is true for $k = 1$, by Lemma 2 and the definition of the function $\tilde{Z}_{M,1}(s)$. Assume that (72) holds true for all $1 \leq j \leq k$. By computing the derivative of formula (72), we get the expression

$$(Z_M H_M)^{(k+1)}(s) = (f_M(s))^k \eta_M(s) K_M^{-1}(s) Z_M(1-s) \prod_{i=1}^k \tilde{Z}_{M,i}(1-s) \cdot \left[k \frac{f'_M}{f_M}(s) + \frac{\eta'_M}{\eta_M}(s) - \frac{K'_M}{K_M}(s) - \sum_{i=0}^k \frac{\tilde{Z}'_{M,i}}{\tilde{Z}_{M,i}}(1-s) \right] \quad (73)$$

$$= (f_M(s))^{k+1} \eta_M(s) K_M^{-1}(s) Z_M(1-s) \prod_{i=1}^{k+1} \tilde{Z}_{M,i}(1-s), \quad (74)$$

where we have used the definition of $\tilde{Z}_{M,k+1}$ (71) to go from (73) to (74). ■

Lemma 18 *For $j \geq 1$, let $Z_{M,j}(s) := \tilde{Z}_{M,j}(s) - 1$. For small $\delta > 0$ and $\delta_1 > 0$, let σ_1 be a real number such that $\sigma_1 \geq 1/2 + \delta_1 > 1/2$ and $(\sigma_1 \pm iT)$ is away from circles of a fixed, small radius $\delta > 0$, centered at integers. Then for $k = 0, 1$*

$$Z_{M,j}^{(k)}(\sigma_1 \pm iT) = O\left(\frac{(T \log T)^{2-2\sigma_1} \log^k T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \rightarrow \infty, \quad (75)$$

and

$$\frac{\tilde{Z}'_{M,j}}{\tilde{Z}_{M,j}}(\sigma_1 \pm iT) = O\left(\frac{(T \log T)^{2-2\sigma_1} \log T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \rightarrow \infty. \quad (76)$$

Proof. We will prove the statement by induction in $j \geq 1$. When $j = 1$, we use formula (32), which we differentiate, use the bound on the growth of the derivative of the digamma function (see formula 6.4.12. in [1]) and the bound (25) with $k = 0$ or $k = 1$. These computations, which are elementary, allows one to prove (75) for $\sigma_1 \geq 1/2 + \delta_1 > 1/2$ in the case when $j = 1$. In addition, by writing

$$\frac{\tilde{Z}'_{M,1}}{\tilde{Z}_{M,1}}(\sigma_1 \pm iT) = \frac{Z'_{M,1}(\sigma_1 \pm iT)}{1 + Z_{M,1}(\sigma_1 \pm iT)} = O\left(\frac{(T \log T)^{2-2\sigma_1} \log T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \rightarrow \infty.$$

With all this, we have proved (76) for $j = 1$.

Assume now that (75) and (76) hold true for all $1 \leq m \leq j$. Then, by (71) we get

$$1 + Z_{M,j+1}(s) = \tilde{Z}_{M,j+1}(s) = 1 + Z_{M,j}(s) + \frac{1}{f_M(s)} \left(\frac{f'_M}{f_M}(s) - \frac{\tilde{Z}'_{M,k+1}}{\tilde{Z}_{M,j}}(s) \right).$$

Therefore, by the inductive assumption on $\tilde{Z}'_{M,j}/\tilde{Z}_{M,j}$ and $Z_{M,j}$, we have

$$Z_{M,j+1}(\sigma_1 \pm iT) = O\left(\frac{(T \log T)^{2-2\sigma_1} \log T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \rightarrow \infty.$$

In other words, (75) holds true with $m = j + 1$. In addition,

$$\frac{\tilde{Z}'_{M,j+1}}{\tilde{Z}_{M,j+1}}(\sigma_1 \pm iT) = \frac{Z'_{M,j+1}(\sigma_1 \pm iT)}{1 + Z_{M,j+1}(\sigma_1 \pm iT)} = O\left(\frac{(T \log T)^{2-2\sigma_1} \log T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \rightarrow \infty.$$

In other words, (76) holds true for $m = j + 1$, which completes the proof. ■

For any integer $k \geq 2$, let us define

$$a_{M,k} := (-1)^{k-1} a_M \log^{k-1} A_M$$

where we set $a_{M,1} := a_M$. The analogue of the function $X_M(s)$, defined by (38), is

$$X_{M,k}(s) := \frac{A_M^s}{a_{M,k}} (Z_M H_M)^{(k)}(s). \quad (77)$$

where, of course, $X_{M,1}(s) = X_M(s)$.

Lemma 19 *For any integer $k \geq 1$, there exists constants $\sigma_k > 1$ and $0 < c_{\Gamma,k} < 1$ such that for all $\sigma = \operatorname{Re}(s) \geq \sigma_k$,*

$$X_{M,k}(s) = 1 + O(c_{\Gamma,k}^\sigma) \neq 0 \quad \text{as } \sigma \rightarrow +\infty.$$

Proof. For $k = 1$, the statement is Lemma 9. Furthermore, from the proof of Lemma 9 and the definition of constants A_M and $a_{M,1}$, we see that, for $\operatorname{Re}(s) \gg 0$

$$\mathcal{D}_1(s) := \sum_{\{P\} \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s} = \frac{a_{M,1}}{A_M^s} \left(1 + O\left(\frac{1}{A_{\Gamma,1}^{\operatorname{Re}(s)}}\right) \right) \quad \text{as } \operatorname{Re}(s) \rightarrow +\infty,$$

for some constant $A_{\Gamma,1} > 1$. Therefore,

$$(Z_M H_M)'(s) = Z_M(s) H_M(s) \mathcal{D}_1(s), \quad (78)$$

where $\mathcal{D}_1(s)$ is a Dirichlet series, converging absolutely for $\operatorname{Re} s > \sigma_1$, for sufficiently large σ_1 , with the leading term equal to $a_{M,1} \cdot A_M^{-s}$ as $\operatorname{Re}(s) \rightarrow +\infty$.

Let us define, for $k \geq 1$ and $\operatorname{Re}(s) \gg 0$

$$(Z_M H_M)^{(k)}(s) = Z_M(s) H_M(s) \mathcal{D}_k(s).$$

We claim that $\mathcal{D}_k(s)$ is a Dirichlet series with the leading term equal to $a_{M,k} \cdot A_M^{-s}$ as $\operatorname{Re}(s) \rightarrow +\infty$. The statement is obviously true for $k = 1$. We assume that it is true for all $1 \leq j \leq k$. Differentiating the above equation for $\operatorname{Re}(s) \gg 0$, we get, from (78)

$$(Z_M H_M)^{(k+1)}(s) = (Z_M H_M)(s) (\mathcal{D}_k(s) \mathcal{D}_1(s) + \mathcal{D}_k'(s)) := (Z_M H_M)(s) \mathcal{D}_{k+1}(s).$$

Since $\mathcal{D}_k(s)$ is a Dirichlet series with the leading term equal to $a_{M,k} \cdot A_M^{-s}$ as $\operatorname{Re} s \rightarrow +\infty$, we see that $\mathcal{D}_k(s) \mathcal{D}_1(s)$ is a Dirichlet series with leading term equal to $(a_{M,k} a_{M,1}) \cdot (A_M^2)^{-s}$ as $\operatorname{Re} s \rightarrow +\infty$, while $\mathcal{D}_k'(s)$ is a Dirichlet series with the leading term equal to $(-a_{M,k} \log(A_M)) \cdot A_M^{-s}$ as $\operatorname{Re} s \rightarrow +\infty$. By the definition of A_M , it is obvious that $A_M > 1$, hence $(A_M^2)^{-s} < A_M^{-s}$ for $\operatorname{Re}(s) \gg 0$. Therefore, $\mathcal{D}_{k+1}(s)$ is a Dirichlet series with the leading term equal to $(-a_{M,k} \log(A_M)) \cdot A_M^{-s}$ as $\operatorname{Re} s \rightarrow +\infty$. By the definition of coefficients $a_{M,k}$ we have that $a_{M,k+1} = -a_{M,k} \log(A_M)$, hence the inductive proof is complete.

For $\operatorname{Re}(s) = \sigma \gg 0$, we may write

$$(Z_M H_M)^{(k)}(s) = Z_M(s) H_M(s) \frac{a_{M,k}}{A_M^s} (1 + O(A_{\Gamma,k}^{-\sigma})) \quad \text{as } \operatorname{Re}(s) \rightarrow \infty.$$

Since

$$Z_M(s) = 1 + O\left(\frac{1}{N(P_{00})^{\operatorname{Re}(s)}}\right) \text{ and } H_M(s) = 1 + O\left(\frac{1}{r_1^{2\operatorname{Re}(s)}}\right) \text{ as } \operatorname{Re}(s) \rightarrow \infty,$$

there exists $\sigma_k \geq 1$ and a constant $C_{\Gamma,k} > 1$ such that for $\operatorname{Re} s > \sigma_k$, we have

$$(Z_M H_M)^{(k)}(s) = \frac{a_{M,k}}{A_M^s} \left[1 + O\left(\frac{1}{C_{\Gamma,k}^{\operatorname{Re}(s)}}\right) \right] \text{ as } \operatorname{Re}(s) \rightarrow \infty.$$

Setting $c_{\Gamma,k} = 1/C_{\Gamma,k}$ completes the proof. ■

Lemma 20 *For arbitrary $\epsilon > 0$, $t \geq 1$ and $\sigma_2 \geq 1$ such that $-\sigma_2$ is not a pole of $(Z_M H_M)$ we have, for any positive integer k ,*

$$(Z_M H_M)^{(k)}(\sigma + it) = \begin{cases} O(\exp \epsilon t), & \text{for } \frac{1}{2} \leq \sigma \leq \sigma_0 \\ O(\exp(1/2 - \sigma + \epsilon)t) & \text{for } -\sigma_2 \leq \sigma < 1/2, \end{cases}$$

as $t \rightarrow \infty$.

Proof. When $k = 1$, the statement is proved in Lemma 13. Assume that the statement of Lemma holds for an integer $k \geq 1$. Then for $1/2 \leq \operatorname{Re}(s) = \sigma \leq \sigma_0$ the Cauchy integral formula yields

$$(Z_M H_M)^{(k+1)}(s) = \frac{1}{2\pi i} \int_C \frac{(Z_M H_M)^{(k)}(z)}{(z-s)^2} dz$$

where C is a circle of a small, fixed radius $r < \epsilon$, centered at s . Using the inductive assumption on $(Z_M H_M)^{(k)}(z)$, we then get the bounds

$$(Z_M H_M)^{(k+1)}(\sigma + it) = O(\exp((r + \epsilon)t)/r) = O(\exp(2\epsilon t)),$$

for $1/2 \leq \sigma \leq \sigma_0$ and $t \geq 1$. This proves the first part of Lemma for $(Z_M H_M)^{(k+1)}(z)$, hence, the first part of the Lemma holds true for all $k \geq 1$.

In the case when $\sigma < 1/2$, we employ the functional equation (72) for $(Z_M H_M)^{(k)}$ to deduce that

$$\begin{aligned} |(Z_M H_M)^{(k+1)}(-\sigma_2 + it)| &= |(Z_M H_M)^{(k)}(-\sigma_2 + it)| \\ &\cdot \left| \left[k \frac{f'}{f}(-\sigma_2 + it) + \frac{\eta'_M}{\eta_M}(-\sigma_2 + it) - \frac{K'_M}{K_M}(-\sigma_2 + it) - \sum_{i=0}^k \frac{\tilde{Z}'_{M,i}}{\tilde{Z}_{M,i}}(1 + \sigma_2 - it) \right] \right|. \end{aligned}$$

Since $\sigma_2 \geq 1$, we have $Z'_M/Z_M(1 + \sigma_2 - it) = O(1)$ as $t \rightarrow +\infty$. Furthermore, formula (76) and the same computations as in the proof of Lemma 13 imply that

$$k \frac{f'}{f}(-\sigma_2 + it) + \frac{\eta'_M}{\eta_M}(-\sigma_2 + it) - \frac{K'_M}{K_M}(-\sigma_2 + it) - \sum_{i=0}^k \frac{\tilde{Z}'_{M,i}}{\tilde{Z}_{M,i}}(1 + \sigma_2 - it) = O(t) \text{ as } t \rightarrow \infty,$$

since the leading term in the above expression is $\operatorname{vol}(M)(1/2 + \sigma_2 - it) \tan(\pi(1/2 + \sigma_2 - it))$. By the inductive assumption on $(Z_M H_M)^{(k)}(-\sigma_2 + it)$, we get

$$\left| (Z_M H_M)^{(k+1)}(-\sigma_2 + it) \right| = O \left(\exp \left(\left(\frac{1}{2} + \sigma_2 + \epsilon \right) \text{vol}(M)t \right) \right), \quad \text{as } t \rightarrow \infty.$$

As in the proof of Lemma 13, one applies the Phragmen-Lindelöf theorem to the function $(Z_M H_M)^{(k+1)}$ in the open sector bounded by the lines $\text{Im}(s) = 1$, $\text{Re}(s) = -\sigma_2$ and $\text{Re}(s) = 1/2$. As a result, the proof of the second part of the Lemma is complete for $(Z_M H_M)^{(k+1)}$. ■

8.2 Distribution of zeros of $(Z_M H_M)^{(k)}$

The following theorem is the analogue of the Main Theorem for zeros of higher derivatives of $(Z_M H_M)$.

Theorem 21 *With the notation as above, the following statements are true for any positive integer k .*

a) *For $\sigma < 1/2$, there exist $t_0 > 0$ such that $(Z_M H_M)^{(k)}(\sigma + it) \neq 0$ for all $|t| > t_0$.*

b)

$$N_{\text{ver}}(T; (Z_M H_M)^{(k)}) = N_{\text{ver}}(T; (Z_M H_M)') + o(T) \quad \text{as } T \rightarrow \infty. \quad (79)$$

c)

$$\begin{aligned} N_{\text{hor}}(T; (Z_M H_M)^{(k)}) &= N_{\text{hor}}(T; (Z_M H_M)') + \frac{(k-1)T}{2\pi} [\log(T \cdot \text{vol}(M)) - 1] \\ &\quad - \frac{T}{2\pi} \log((k-1) \log A_M) + o(T) \quad \text{as } T \rightarrow \infty. \end{aligned} \quad (80)$$

Proof. We first outline the proof of part (a). For $k \geq 2$, $\sigma < 1/2$ and $s = \sigma \pm iT$ equation (72) yields

$$\begin{aligned} \frac{(Z_M H_M)^{(k)}}{(Z_M H_M)^{(k-1)}}(s) &= \log \left((Z_M H_M)^{(k-1)}(s) \right)' \\ &= (k-1) \frac{f'}{f}(s) + \frac{\eta'_M}{\eta_M}(s) - \frac{K'_M}{K_M}(s) - \frac{Z'_M}{Z_M}(1-s) - \sum_{i=1}^{k-1} \frac{\tilde{Z}'_{M,i}}{\tilde{Z}_{M,i}}(1-s) \end{aligned} \quad (81)$$

We now apply (76) with $\sigma_1 = 1 - \sigma > 1/2$ and (25) to deduce that

$$\frac{Z'_M}{Z_M}(1-s) + \sum_{i=1}^{k-1} \frac{\tilde{Z}'_{M,i}}{\tilde{Z}_{M,i}}(1-s) = O \left(\frac{(T \log T)^{2\sigma} \log T}{(1/2 - \sigma)} \right) \quad \text{as } T \rightarrow \infty.$$

Since $\text{Re}(\eta'_M/\eta_M(\sigma \pm iT)) = -\text{vol}(M)t + O(\log t)$ and $K'_M/K_M(\sigma \pm iT) = O(\log t)$ as $t \rightarrow +\infty$, we immediately deduce from (81) that

$$\text{Re} \left(-\frac{(Z_M H_M)^{(k)}}{(Z_M H_M)^{(k-1)}}(\sigma \pm it) \right) = \text{vol}(M)t + O \left(\max \left\{ \log t, \frac{(t \log t)^{2\sigma} \log t}{(1/2 - \sigma)} \right\} \right) \quad \text{as } t \rightarrow +\infty,$$

for any $\sigma < 1/2$. This proves part a).

The proof of parts b) and c) closely follows lines of the proof of parts b) and c) of the Main Theorem. We fix a large positive number T and choose number T' to be a bounded distance from T

such that T' is distinct from the imaginary part of any zero of $Z_M H_M$. We fix a number $a \in (0, 1/2)$ and use part a) of the Theorem to choose $t_0 > 0$ to be the number such that $(Z_M H_M)^{(k)}(\sigma + it) \neq 0$ for all $\sigma \leq a$ and $|t| > t_0$. Let σ_0 be a constant such that $\sigma_0 \geq \max\{\sigma'_0, \sigma_k\}$, where σ'_0 is defined in Lemma 1 and σ_k is defined in Lemma 19.

We apply Littlewood's theorem to the function $X_{M,k}(s)$, defined by (77) which is holomorphic in the rectangle $R(a, T')$ with vertices $a + it_0$, $\sigma_0 + it_0$, $\sigma_0 + iT'$, $a + iT'$. The resulting formula is

$$2\pi \sum_{\substack{\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)} \\ t_0 < \gamma^{(k)} < T', \beta^{(k)} > a}} (\beta^{(k)} - a) = \int_{t_0}^{T'} \log |X_{M,k}(a + it)| dt - \int_{t_0}^{T'} \log |X_{M,k}(\sigma_0 + it)| dt \quad (82)$$

$$- \int_a^{\sigma_0} \arg X_{M,k}(\sigma + it_0) d\sigma + \int_a^{\sigma_0} \arg X_{M,k}(\sigma + iT') d\sigma \quad (83)$$

$$= I_{1,k} + I_{2,k} + I_{3,k} + I_{4,k}, \quad (84)$$

where $\rho^{(k)}$ denotes the zero of $(Z_M H_M)^{(k)}$. By the choice of t_0 , the sum on the left-hand side of (82) is actually taken over all zeros $\rho^{(k)}$ of $(Z_M H_M)^{(k)}$ with imaginary part in the interval (t_0, T') .

Trivially, $I_{3,k} = O(1)$ as $T \rightarrow +\infty$. The application of Lemma 19 immediately yields that $I_{2,k} = O(1)$ as $T \rightarrow +\infty$, once we apply the same method as in evaluation of I_2 .

One can follow the steps of the proof that $|\arg X_M(\sigma + iT')| = o(T)$ as $T \rightarrow +\infty$ in the present setting. One uses function $X_{M,k}$ instead of X_M and Lemma 20 instead of Lemma 13. From this, we deduce that $|\arg X_{M,k}(\sigma + iT')| = o(T)$ as $T \rightarrow +\infty$. Therefore, it is left to evaluate $I_{1,k}$.

From definition of $X_{M,k}$, using the functional equation (72) for $(Z_M H_M)^{(k)}$, we get for $k \geq 2$, the expression

$$\begin{aligned} I_{1,k} = & \int_{t_0}^{T'} \log |A_M^{(a+it)} a_{M,k}^{-1}| dt + k \int_{t_0}^{T'} \log |f_M(a + it)| dt + \int_{t_0}^{T'} \log |\eta_M(a + it)| dt + \\ & + \int_{t_0}^{T'} \log |K_M^{-1}(a + it)| dt + \int_{t_0}^{T'} \log |Z_M(1 - a - it)| dt + \sum_{i=1}^{k-1} \int_{t_0}^{T'} \log |1 + Z_{M,i}(1 - a - it)| dt. \end{aligned} \quad (85)$$

By employing equation (75) with $j = 0$, we get

$$\log |1 + Z_{M,i}(1 - a - it)| = O(|Z_{M,i}(1 - a - it)|) = O(t^{2a-1}(\log t)^{2a}) \quad \text{as } t \rightarrow \infty$$

and for all $i = 1, \dots, k-1$. Hence,

$$\int_{t_0}^{T'} \log |1 + Z_{M,i}(1 - a - it)| dt = O((T \log T)^{2a}) \quad \text{as } T \rightarrow \infty$$

and for all $i = 1, \dots, k-1$. Substituting this equation, together with (51), (53), (54) and (56) into (85), we immediately deduce that

$$I_{1,k} = \left(\frac{1}{2} - a\right) \frac{\text{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + k\right) T \log T + C_{M,a,k} T + O((T \log T)^{2a}) \quad \text{as } T \rightarrow \infty,$$

where

$$\begin{aligned} C_{M,a,k} = & 2 \left(a - \frac{1}{2}\right) n_1 \log 2 + a \log A_M - \log |a_{M,k}| \\ & + k(\log(\text{vol}(M)) - 1) + 2a \log \mathfrak{g}_1 - \log |d(1)| - \frac{n_1}{2}(\log \pi + 1). \end{aligned}$$

Combining this equation with the bounds on $I_{2,k}$, $I_{3,k}$ and $I_{4,k}$ and (82), we get

$$2\pi \sum_{\substack{\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)} \\ t_0 < \gamma^{(k)} < T'}} \left(\beta^{(k)} - a\right) = \left(\frac{1}{2} - a\right) \frac{\text{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + k\right) T \log T + C_{M,a,k} T + o(T) \quad \text{as } T \rightarrow \infty. \quad (86)$$

Replacing a by $a/2$ in (86) and subtracting proves part b). Part c) is proved by employing an analogue of equation (62), with β' and ρ' replaced by $\beta^{(k)}$ and $\rho^{(k)}$. ■

Remark 22 The statement of Theorem 21 is true in the case of co-compact Riemann surfaces $\Gamma \backslash \mathbb{H}$ when taking $H_M = 1$ and $A_M = \exp(\ell_{M,0})$ in (79) and (80).

In the case when $\Gamma \backslash \mathbb{H}$ is compact the statement b) of Theorem 21 was announced by Luo in [44], with the weaker error term $O(T)$. As one can see, we put consider effort into the analysis yielding the error term $o(T)$, and the structure of the constant $C_{M,a,k}$ is, in our opinion, fascinating.

Remark 23 From the formula (80) for the horizontal distribution of zeros of $(Z_M H_M)^{(k)}$, we see that the differentiation of $(Z_M H_M)^{(k)}$ increases the sum $N_{\text{hor}}(T; (Z_M H_M)^{(k)})$ by the quantity $[(\text{vol}(M)/2\pi) \cdot T \log T - O(T)]$ as $T \rightarrow \infty$. Hence, after each differentiation, zeros of $(Z_M H_M)'$ move further to the right of $1/2$. Since every zero of $(Z_M H_M)'$ on the line $\text{Re}(s) = 1/2$ (up to finitely many of them) is a multiple zero of Z_M , this result fully supports the "bounded multiplicities conjecture". To recall, the "bounded multiplicities conjecture" asserts that the order of every multiple zero of Z_M is uniformly bounded, or, equivalently, that the dimension of every eigenspace associated to the discrete eigenvalue of the Laplacian on M is uniformly bounded, with a bound depending solely upon M .

9 Concluding Remarks

9.1 Revisiting Weyl's law

Weyl's law for an arbitrary finite volume hyperbolic Riemann surface M is the following asymptotic formula, which we quote from [29], p. 466:

$$N_{M,\text{dis}}(T) + N_{M,\text{con}}(T) = \frac{\text{vol}(M)}{4\pi} T^2 - \frac{n_1}{\pi} T \log T + \frac{n_1 T}{\pi} (1 - \log 2) + O(T/\log T) \quad \text{as } T \rightarrow \infty, \quad (87)$$

where

$$N_{M,\text{dis}}(T) = \#\{s = 1/2 + it | Z_M(s) = 0 \text{ and } 0 \leq t \leq T\}$$

and

$$N_{M,\text{con}}(T) = \frac{1}{4\pi} \int_{-T}^T \frac{-\phi'_M}{\phi_M}(1/2 + it) dt.$$

For a proof of Weyl's law, we refer to Theorem 2.28 on page 466 of [29] and Theorem 7.3 of [60].

The term $N_{M,\text{dis}}(T)$ counts the number of zeros of the Selberg zeta function $Z_M(s)$ on the critical line $\text{Re}(s) = 1/2$, whereas the term $N_{M,\text{con}}(T)$ is related to the number of zeros of $Z_M(s)$ off the critical line but within the critical strip. In the following proposition, we will relate the counting function $N_{\text{ver}}(T; \phi_M)$ with the function $N_{M,\text{con}}(T)$, showing that the constant \mathfrak{g}_1 appears in the resulting asymptotic formula.

Proposition 24 *There exists a sequence $\{T_n\}$ of positive numbers tending toward infinity such that, with the notation as above, we have the asymptotic formula*

$$N_{\text{ver}}(T_n; \phi_M) = N_{M,\text{con}}(T_n) - \frac{\log \mathfrak{g}_1}{\pi} T_n + O(\log T_n) \text{ as } n \rightarrow \infty.$$

Proof. Let $R(T)$ denote the rectangle with vertices $1/2 - iT$, $\sigma'_0 - iT$, $\sigma'_0 + iT$, $1/2 + iT$, where $\sigma'_0 > \sigma_0$, where σ_0 is defined in section 1.4.. Therefore, the series

$$\frac{H'_M}{H_M}(s) = \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s}$$

converges uniformly and absolutely for $\text{Re } s \geq \sigma'_0$, and all zeros of ϕ_M with real part greater than $1/2$ lie inside $R(T)$. Recall that the zeros of ϕ_M appear in pairs of the form ρ and $\bar{\rho}$. As a result, the proposition will follow by studying the expression

$$2N_{\text{ver}}(T; \phi_M) = \frac{1}{2\pi i} \int_{R(T)} \frac{\phi'_M}{\phi_M}(s) ds.$$

Let us write

$$2N_{\text{ver}}(T; \phi_M) = -\frac{1}{2\pi} \int_{-T}^T \frac{\phi'_M}{\phi_M}\left(\frac{1}{2} + it\right) dt + \frac{1}{2\pi} \int_{-T}^T \frac{\phi'_M}{\phi_M}(\sigma'_0 + it) dt + I_1(T) + I_2(T)$$

where I_1 and I_2 denote the integrals along the horizontal lines which bound $R(T)$. Theorem 7.1 on page 412 of [36] proves that ϕ_M is of regularized product type with order $M = 0$. As a result, from Chapter 1, section 4 of [37], we have the existence of a sequence of real numbers $\{T_n\}$ tending to infinity such that

$$I_1(T_n) = O(\log T_n) \text{ and } I_2(T_n) = O(\log T_n) \text{ when } n \rightarrow \infty,$$

so then

$$2N_{\text{ver}}(T_n; \phi_M) = -\frac{1}{2\pi} \int_{-T_n}^{T_n} \frac{\phi'_M}{\phi_M} \left(\frac{1}{2} + it \right) dt + \frac{1}{2\pi} \int_{-T_n}^{T_n} \frac{\phi'_M}{\phi_M} (\sigma'_0 + it) dt + O(\log T_n) \quad \text{when } n \rightarrow \infty.$$

Using the notation as above, we now write

$$\begin{aligned} \int_{-T_n}^{T_n} \frac{\phi'_M}{\phi_M} (\sigma'_0 + it) dt &= \int_{-T_n}^{T_n} \frac{H'_M}{H_M} (\sigma'_0 + it) dt + \int_{-T_n}^{T_n} \frac{K'_M}{K_M} (\sigma'_0 + it) dt \\ &= \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^{\sigma'_0}} \int_{-T_n}^{T_n} \frac{dt}{q_i^{it}} - 4T \log \mathfrak{g}_1 + n_1 \int_{-T_n}^{T_n} \left(\frac{\Gamma'}{\Gamma} \left(\sigma'_0 + it - \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} (\sigma'_0 + it) \right) dt. \end{aligned}$$

Interchanging the sum and the integral above is justified by the fact that the series defining $H'_M/H_M(s)$ converges absolutely and uniformly for $\text{Re}(s) > \sigma_0$. Furthermore, we also have that

$$\sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^{\sigma'_0}} \int_{-T_n}^{T_n} \frac{dt}{q_i^{it}} = O(1) \quad \text{as } n \rightarrow \infty,$$

Using the series representation of the digamma function, we see that

$$\frac{\Gamma'}{\Gamma} \left(\sigma'_0 + it - \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} (\sigma'_0 + it) = \sum_{k=0}^{\infty} \frac{-1/2}{(k + \sigma'_0 + it - 1/2)(k + \sigma'_0 + it)}$$

The series on the right converges uniformly in $t \in [-T_n, T_n]$, hence

$$\int_{-T_n}^{T_n} \left(\frac{\Gamma'}{\Gamma} \left(\sigma'_0 + it - \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} (\sigma'_0 + it) \right) dt = \sum_{k=0}^{\infty} \log \left[1 - \frac{iT_n}{(k + \sigma'_0)(k + \sigma'_0 - 1/2) + T_n^2 + iT_n/2} \right].$$

Since

$$\sum_{k=0}^{\infty} \left| \log \left[1 - \frac{iT}{(k + \sigma'_0)(k + \sigma'_0 - 1/2) + T^2 + iT/2} \right] \right| \ll |T| \int_0^{\infty} \frac{dx}{x^2 + T^2},$$

we get that

$$\int_{-T_n}^{T_n} \left(\frac{\Gamma'}{\Gamma} \left(\sigma'_0 + it - \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} (\sigma'_0 + it) \right) dt = O(1) \quad \text{as } n \rightarrow \infty.$$

With all this, the proof of the Proposition is complete. \blacksquare

Remark 25 The above proposition shows that the term $-(\log \mathfrak{g}_1/\pi)T$ measures the discrepancy between the number of zeros of ϕ_M with real part greater than $1/2$, meaning $N_{\text{ver}}(T; \phi_M)$, and the quantity $N_{M, \text{con}}(T)$, appearing in the classical version of the Weyl's law.

Furthermore, one can restate Proposition 24 as the relation representing Weyl's law

$$N_{\text{ver}}(T_n; Z_M H_M) = \frac{\text{vol}(M)}{4\pi} T_n^2 - \frac{n_1}{\pi} T_n \log T_n + \frac{T_n}{\pi} (n_1(1 - \log 2) - \log \mathfrak{g}_1) + O(T_n / \log T_n), \quad (88)$$

as $n \rightarrow \infty$.

A direct consequence of the relation (88), Main Theorem and Theorem 21 is the following reformulation of the Weyl's law:

Corollary 26 *There exist a sequence $\{T_n\}$ of positive real numbers tending to infinity such that, for every positive integer k*

$$N_{\text{ver}}(T_n; Z_M H_M) = N_{\text{ver}}(T_n; (Z_M H_M)^{(k)}) - \frac{n_1}{\pi} T_n \log T_n + \frac{T_n}{2\pi} (2n_1 + \log A_M) + o(T_n), \text{ as } n \rightarrow \infty.$$

Remark 27 An interpretation of the constant $\log \mathfrak{g}_1$, similar to the one derived in Proposition 24 is obtained in [20], formula (3.4.15) on page 155, where it is shown, in our notation, that

$$\log \mathfrak{g}_1 = \lim_{x \rightarrow \infty} \lim_{y \rightarrow \infty} \left[2x \int_0^\infty \frac{N_{M, \text{con}}(t) - N_{\text{ver}}(t; \phi_M)}{t} \left(\frac{1}{t^2 + x^2} - \frac{1}{t^2 + y^2} \right) dt \right].$$

A geometric interpretation of the constant \mathfrak{g}_1 , in the case when the surface has one cusp \mathfrak{a} is derived on page 49 of [34]. In that case, \mathfrak{g}_1^{-1} is the radius of the largest isometric circle arising in the construction of the standard polygon for the group $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$, where $\sigma_{\mathfrak{a}}$ denotes the scaling matrix of the cusp \mathfrak{a} .

As stated in the introduction, we do not know of a spectral or geometric interpretation of the constant \mathfrak{g}_2 , besides the trivial one which realizes \mathfrak{g}_2 as the second largest denominator of the Dirichlet series portion of the scattering determinant. Therefore, we view Corollary 26 as giving rise to a new spectral invariant.

9.2 A comparison of counting functions

In this section we will prove (12) for k th derivatives, where $k \geq 1$. In effect, it is necessary to recall results from [29], translate the notation in [29] to the notation in the present paper, then combine the result with (5) and parts (c) of the Main Theorem and Theorem 21.

Theorem 2.22 on page 456 of [29] states the asymptotic relation

$$N_{\text{hor}}(T; H_M) = \frac{n_1}{2} \cdot \frac{T \log T}{2\pi} + \frac{T}{2} b_6 + O(\log T) \text{ as } T \rightarrow \infty, \quad (89)$$

where the constant b_6 is given on the first line on page 448, by

$$b_6 = -\frac{n_1}{2\pi} - \frac{1}{\pi} \log |b_2|.$$

Note that in [29], the author counts the zeros of H_M in both the upper and lower half-planes, whereas the counting function $N_{\text{hor}}(T; H_M)$ only considers those zeros in the upper half-plane. Recall that

the zeros and poles of H appear symmetrically about the real axis. As a result, the relation (89) differs from Theorem 2.22, page 456 of [29], by a factor of two. We now relate the constant b_2 in (89) to the notation we employed above.

Equation (2.15) on page 445 of [29] gives

$$\begin{aligned} f_M(s) &= \mathfrak{g}_1^{2s-1} \varphi(s) = \mathfrak{g}_1^{2s-1} \pi^{\frac{n_1}{2}} \left(\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^{n_1} \cdot \sum_{n=1}^{\infty} \frac{d(n)}{\mathfrak{g}_n^{2s}} \\ &= \pi^{\frac{n_1}{2}} \left(\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^{n_1} \mathfrak{g}_1^{2s-1} \frac{d(1)}{\mathfrak{g}_1^{2s}} \left(1 + \sum_{n=2}^{\infty} \frac{a(n)}{r_n^{2s}} \right) \\ &= b_2 \left(\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^{n_1} \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{Q_n^{2s}}, \end{aligned}$$

where the last line is in the notation of page 446, line 5 of [29] with $m = 0$; see also equation (2.16) on page 445 of [29]. When comparing with our notation, as defined in section 1.3 and section 2.3, we have that $Q_n = r_n$ and, more importantly,

$$b_2 = \pi^{\frac{n_1}{2}} \mathfrak{g}_1^{-1} d(1). \quad (90)$$

By substituting (90) into (89), we are able to rewrite Theorem 2.22 on page 456 of [29] as

$$N_{\text{hor}}(T; H_M) = \frac{n_1}{2} \cdot \frac{T \log T}{2\pi} - \frac{T}{2\pi} \left(\frac{n_1}{2} + \frac{n_1}{2} \log \pi + \log |d(1)| - \log \mathfrak{g}_1 \right) + O(\log T) \quad \text{as } T \rightarrow \infty. \quad (91)$$

Comparing (91) with part (c) of Main Theorem we deduce that

$$N_{\text{hor}}(T; (Z_M H_M)') - N_{\text{hor}}(T; H_M) = \frac{T \log T}{2\pi} + \frac{T}{2\pi} C + o(T) \quad \text{as } T \rightarrow \infty.$$

with

$$C = \frac{1}{2} \log A_M - \log |a_M| + \log \text{vol}(M) - 1.$$

Assume that $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$, so then

$$A_M = e^{\ell_{M,0}} \quad \text{and} \quad a_M = \frac{m_{M,0} \ell_{M,0}}{1 - e^{-\ell_{M,0}}},$$

from which we have that

$$C = \log \left(\frac{2 \text{vol}(M) \sinh(\ell_{M,0}/2)}{e \cdot m_{M,0} \ell_{M,0}} \right).$$

Let \widetilde{M} be any co-compact hyperbolic Riemann surface such that $\text{vol}(\widetilde{M}) = \text{vol}(M)$. Assume that M and \widetilde{M} have systoles of equal length, and the same number of inconjugate classes of systoles. Then, using (5), we arrive at the conclusion that

$$N_{\text{hor}}(T; (Z_M H_M)') - N_{\text{hor}}(T; H_M) = N_{\text{hor}}(T; Z_{\widetilde{M}}') + o(T) \quad \text{as } T \rightarrow \infty, \quad (92)$$

as claimed in (12).

Furthermore, when $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$, comparing (92) with part c) of Theorem 21, for $k \geq 2$ we arrive at

$$\begin{aligned} N_{\text{hor}}(T; (Z_M H_M)^{(k)}) - N_{\text{hor}}(T; H_M) &= N_{\text{hor}}(T; Z'_M) + \\ &+ \frac{(k-1)T}{2\pi} [\log(T \text{vol}(M)) - 1] - \frac{T}{2\pi} \log((k-1)\ell_{M,0}) + o(T) \quad \text{as } T \rightarrow \infty. \end{aligned}$$

Then, from part c) of Theorem 21 applied to the zeta function $Z_{\widetilde{M}}$ we deduce

$$N_{\text{hor}}(T; (Z_M H_M)^{(k)}) - N_{\text{hor}}(T; H_M) = N_{\text{hor}}(T; (Z_{\widetilde{M}})^{(k)}) + o(T) \quad \text{as } T \rightarrow \infty.$$

This proves (12) for higher derivatives.

We find the comparison of counting functions, as summarized in (12) fascinating, especially since the coefficients in the asymptotic expansions in (89) and part (c) of the Main Theorem are somewhat involved and dissimilar from other known asymptotic expansions.

9.3 Concluding remarks

We find the examples provided by studying $\Gamma_0^+(5)$ and $\Gamma_0^+(6)$ particularly fascinating. In a forthcoming article [38] with Holger Then, we further study various properties of the distribution of eigenvalues on $\Gamma_0^+(N)$, for a squarefree, positive integer N . In particular, we prove the groups $\Gamma_0^+(5)$ and $\Gamma_0^+(6)$ have the same signature yet the coefficient of T in the Weyl's law expansion differs. As stated above, we view this example as being in support of the Phillips-Sarnak philosophy since the examples show that the distribution of eigenvalues depends on more data than just the signature of the group.

In [11] the authors defined 213 genus zero subgroups of which 171 are associated to “Moonshine”. It would be interesting to compute the invariant A_M for each of these groups to see if further information regarding the groups, possibly related to “moonshine”, is uncovered.

Is it possible to explicitly determine an example of a surface where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$? More generally, one could study the set of such surfaces, as a subset of moduli space. Is the set of surfaces where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$ a connected subset of moduli space, or are there several components? Is there another characterization of surfaces where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$? Many other basic questions can be easily posed, and we find these problems very interesting.

In general, one wonders if there is another interpretation for surfaces such that $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. J. Keating pointed out that in many classical dynamical systems, one can have complex periodic orbits. In the setting of the Selberg trace formula, we were referred to [9] and [10]. Perhaps the inequality $e^{\ell_{M,0}} \neq (\mathfrak{g}_2/\mathfrak{g}_1)^2$ is related to the existence of complex orbits as in [9] and [10]. It was also suggested to us that a connection may exist with Wigner delay times associated to classical scattering problems, which is related to Keating's remarks. We thank Keating for allowing us to include his speculative comments in this article, who wishes to emphasize that his suggestion should not be viewed as a thought-out conjecture. Nonetheless, the authors believe it is important to include his intuitive comments.

In [2], the authors determined the asymptotic behavior of Selberg's zeta function through degeneration up to the critical line. It would be interesting to study the asymptotic behavior of the zeros of the derivative of Selberg's zeta function through degeneration, either in moduli space or through elliptic degeneration.

To come full circle, we return to the setting of the Riemann zeta function and speculate if one can attempt to mimic results which follow from the Levinson-Montgomery article [42]. Specifically, we recall, as stated in the Introduction, that Levinson used results from the distribution of zeros of

ζ'_Q to prove that more than $1/3$ of the zeros of the Riemann zeta function lie on the critical line. Can one follow a similar investigation in the setting of the Selberg zeta function? To do so, we note that a starting point would be to establish an analogue of the approximate functional equation for the Selberg zeta function. In the case when M is compact, the Riemann hypothesis is known for the Selberg zeta function, but if M is not compact, then one is facing the ideas behind the Phillips-Sarnak philosophy. Results in this direction would be very significant, and we plan to undertake the project in the near future.

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